We continue on with the root space decomposition.

We will write $\mathfrak{g}_{\pm\alpha}$ for the two-dimensional real vector space whose complexification is $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\pm\alpha}$ so that the root space decomposition reads:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} m_a \mathfrak{g}_{\pm\alpha}$$

Theorem 1. *$m_\alpha = 1$ for all $\alpha \in R$. In other words each weight $\pm\alpha$ which occurs in the root space decomposition (*) occurs with multiplicity* 1.

If $\alpha \in R$, and if $k\alpha \in R$, then $k = \pm 1$.

The proof from Hsiang is rather spectacular.

Step 1. Classify all rank 1 compact connected Lie groups. The complete list is , S^1, $SU(2)$ and $SO(3)$.

Step 2. Observe that for each $\alpha \in R$ the kernel, $ker(\alpha)$ defines a subtorus T_α of codimension 1,

Step 3. Compute that the centralizer $Z(T_\alpha)$ (the group of all elements g such that $gt = tg$ for all $t \in T_\alpha$) is a closed connected Lie subgroup $G_\alpha \subset G$ whose Lie algebra is $\mathfrak{t} \oplus \bigoplus_{\beta \in R(\alpha)} \mathfrak{g}_{\pm\beta}$, the big sum $\bigoplus$ being over the set $R(\alpha) = R \cap \mathbb{Z}\alpha$ of all roots β which are a nonzero multiple of α..

Step 4. Observe that the quotient group G_α/T_α is a closed compact Lie group of rank 1, since its maximal torus is T/T_α.

Step 5. Conclusion! : By Step 1, the Lie algebra of G_α/T_α agrees with $su(2)$, and in particular has dimension 3. It follows that only a single pair $\pm\alpha$ can occur, and that with multiplicity 1.

QED, modulo proving the pieces 1-3.

Step 1. Takes the most work. Hang on!

Step 2. Recall that we can also view roots (and weights) as homomorphisms $T \to S^1$. As such, their kernels are closed subtorii, of codimension 1, since they are onto.

Step 3. On the Lie algebra level, the centralizer of a subalgebra $\mathfrak{t}_\alpha$ consists of all those elements X which commute with every element of $\mathfrak{t}_\alpha$. We first compute that the Lie algebra centralizer of $\mathfrak{t}_\alpha$ is $\mathfrak{t} \oplus \bigoplus_{\beta \in R(\alpha)} \mathfrak{g}_{\pm\beta}$. Indeed, if $\beta = k\alpha$ then we have that $[\mathfrak{g}_\beta, \mathfrak{t}_\alpha] = 0$ since any element of $exp(H) \in T_\alpha$ acts on $\mathfrak{g}_\beta$ by multiplication by $Z \mapsto exp(i2\pi k\alpha(H))Z$ and since $\alpha(X) = 0$, the action is by the identity. Differentiating we see that get $[X, \mathfrak{t}_\alpha] = 0$. We also have $[\mathfrak{t}, \mathfrak{t}_\alpha] \subset [\mathfrak{t}, \mathfrak{t}] = 0$. This proves that on the Lie algebra level $\mathfrak{t} \oplus \bigoplus_{\beta \in R(\alpha)} \mathfrak{g}_{\pm\beta}$ centralizes. One checks that if β is not an element of $R(\alpha)$ then the generic element of T_α rotates $\mathfrak{g}_{\pm\beta}$, so that $[\mathfrak{t}_\alpha, \mathfrak{g}_\beta] \neq 0$. This shows that the Lie sub-algebra $\mathfrak{t} \oplus \bigoplus_{\beta \in R(\alpha)} \mathfrak{g}_{\pm\beta}$ is the centralizer. Exponentiating we get the Lie group G_α. Finally, a bit of topological work, described nicely in Hsiang, shows that the centralizer of any subtorus, and in particular of T_α is a closed connected Lie subgroup, here denoted G_α.

Step 4. General properties of Lie groups yield that if $N \subset H$ is a closed normal subgroup of a closed connected Lie group H, then the Lie subalgebra $\mathfrak{n} \subset \mathfrak{h}$ is a Lie ideal $[\mathfrak{n}, \mathfrak{h}] \subset \mathfrak{n}$ and G/N is a closed connected Lie group with Lie algebra $\mathfrak{h}/\mathfrak{n}$. Consequently G_α/T_α is a closed connected Lie group with Lie algebra $\mathfrak{t}/\mathfrak{t}_\alpha \oplus \bigoplus_{\beta \in R(\alpha)} \mathfrak{g}_{\pm\beta}$. But $\mathfrak{t}/\mathfrak{t}_\alpha$ is one dimensional and acts nontrivially on all the $\mathfrak{g}_{\pm k\alpha}$, $k\alpha \in R(\alpha)$, showing that the rank of this Lie group is 1, with the Lie algebra of the maximal torus being $\mathfrak{t}/\mathfrak{t}_\alpha$, and showing that $R(\alpha) = \{\pm\alpha\}$.

The big step, step 1.

If $dim(G) = 1$ we are done, G is a circle.

Otherwise, consider the weight space decompostion of $\mathfrak{g}$ which is the decomposition of the restriction to the circle T of G's adjoint action on $\mathfrak{g}$. We have $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus \mathfrak{g}_{\pm\alpha}$, the sum being over weights for the circle and written with multiplicity . Take the lowest occuring weight, call it α_1, with corresponding weight space $\mathfrak{g}_{\pm\alpha_1}$. Observe that $\mathfrak{g}_1 := \mathfrak{t} \oplus \mathfrak{g}_{\pm\alpha_1}$ forms a Lie sub-algebra of $\mathfrak{g}$ isomorphic to $su(2)$. The induced linear isomorphism $su(2) \to \mathfrak{g}_1$ exponentiates to give a Lie group homomorphism $SU(2) \to G$ for which the differential of the image is $\mathfrak{g}1$. (We use $SU(2)$'s simple connectivity to guarantee the existence of the map.) Write $G1$ for the image, a closed Lie subgroup of G, isomorphic to either $SO(3)$ or $SU(2)$. Restricting the adjoint representation to $G_1 \subset G$ then yields a representation of either $SO(3)$ or $SU(2)$ on $\mathfrak{g}$. But $\mathfrak{g}_1$ is a invariant subspace, of this representation of G_1, and so the orthogonal complement $\mathfrak{g}_1^{\perp} = \bigoplus_{\lambda \neq \alpha} \mathfrak{g}_{\pm\lambda}$ is also a representation of G_1. (If the original α_1 occured with multiplicity then their may be a copy of $\mathfrak{g}_{\pm\alpha_1}$ within $\mathfrak{g}_1^{perp}$.) There is no 0 weight for this subrepresentation, since the torus is by assumption dimension 1, and a weight 0 would mean the maximal torus had dimension at least 2. But for $SO(3)$ every representation has a zero weight subspace, arising as it does from a V_{2k}. Consequently $G_1 \neq SO(3)$. This leaves us with $G_1 = SU(2)$, in which case the weight α_1 occuring within $\mathfrak{g}_1$ is weight 2, since that is the root space decomposition of $SU(2)$. Thus all the other weights occurring in the orthogonal representation $\mathfrak{g}_1^{\perp}$ are 2 (allowing for multiplicity) or higher. But this is impossible: any representation of $SU(2)$ is either even, in which case it has a zero weight, which we've already excluded, or odd, in which case it has a weight space of weight 1, which we've also excluded since we took α_1 to be the lowest occurring weight.