0.1. **Weights for a torus T.** We saw that the space of weights for the circle S^1, which is the space $\hat{S}^1$ of its irreps , is isomorphic to $Hom(S^1, S^1)$since all representations are complex 1-dimensional. $Hom(S^1, S^1)$ is in turn naturally isomorphic to the integers $\mathbb{Z}$. (The natural isomorphism is, topologically speaking, the *degree* of the map $S^1 \to S^1$.) Since a torus T is isomorphic to n copies of the circle, it is not a surprise that its space $\hat{T}$ of irreps is $\mathbb{Z}^n$. But it is helpful to see this intrinsically.

The Lie algebra of a torus T, denoted

$$\mathfrak{t} \cong Hom(\mathbb{R}, T)$$

contains an intrinsic lattice

$$ker(exp) \cong Hom(S^1, T) := \Lambda^*.$$

consisting of those one-parameter subgroups $exp(tX)$ that close up after after one period: $exp(X) = 1$. Dually we have that the dual to the Lie algebra $\mathfrak{t}^*$ contains the lattice

$$\Lambda \cong Hom(T, S^1)$$

intrinsically dual to the above lattice. This is the lattice $\hat{T}$ of characters. The pairing between the two lattices, is composition, $Hom(S^1, T) \times Hom(T, S^1) \to Hom(S^1, S^1) \cong \mathbb{Z}$ which gives us composing we get a homomorphism $S^1 \to S^1$, and associated to this homomorphism is an integer, the winding number, thus establishing a pairing

$$Hom(S^1, T) \times Hom(T, S^1) \to Hom(S^1, S^1) \cong \mathbb{Z}.$$

To see how the lattice Λ of characters sits inside the vector space $\mathfrak{t}^*$ we build a character $\chi : T \to S^1$ out of a linear functional $\lambda : \mathfrak{t} \to \mathbb{R}$ by using the exponential:

$$\chi(expX) = e^{i2\pi\lambda(X)}.$$

Note that the requirement $\chi(1) = 1$ requires that $\lambda(X) \in \mathbb{Z}$ whenever $exp(X) = 1$ which is to say, we must insist that λ be in the lattice dual to the lattice $ker(exp)$. For each $\lambda \in \Lambda$ we write V_λ for the corresponding complex one-dimensional weight space.

Weights of a representation. From now on we identify the lattice $\Lambda \subset \mathfrak{t}^*$ with $\hat{T}$, the space of irreducible representations of T. If V is a finite-dimensional representation space for T, then we can break V up into irreducibles: $V = \bigoplus_\chi m_\chi V_\chi$ where the sum is over some finite subset S of the lattice Λ and the integers m_χ are the multiplicities. The "weights" of the representations are the elements of this subset S listed with their multiplicities.

Aside 1. Coordinates. Choosing a basis for $\mathfrak{t}$ consisting of elements in a $\mathbb{Z}$-basis for $ker(exp)$ induces an isomorphism $(ker(exp), \mathfrak{t}) \cong (\mathbb{Z}^n, \mathbb{R}^n)$ and consequently of $T \cong \mathfrak{t}/ker(exp) \cong \mathbb{R}^n/\mathbb{Z}^n$. Then $\Lambda \cong \mathbb{Z}^{n*}$. Written out in in the corresponding angular coordinates $\theta^j(mod 1)$, the characters of T have the form $exp(2\pi i \Sigma n_j \theta^j)$. The form $(t^1, \ldots, t^n) \to \Sigma n_j t^j$ is a weight for T.

Remark 2. The algebraic topological viewpoint on these lattices is:

$$\mathfrak{t} \cong H_1(T; \mathbb{R}) \quad ; \Lambda^* \cong H_1(T; \mathbb{Z}).$$

.

$$\mathfrak{t}^* \cong H^1(T, \mathbb{R}) \quad ; \Lambda \cong H^1(T, \mathbb{Z})$$

.

Take G to be a connected compact Lie group. Fix a maximal torus $T \subset G$. Let V be a representation space for G. By restriction, V is a representation space for T.

Definition 1. *The weights of V are the weights of the restricted representation of T on V. The corresponding joint eigenspaces $V_\lambda, \lambda \in \Lambda$ are called "weight spaces"*

If λ is a weight, $t = \exp(H) \in T$ and $v \in V_\lambda$ then this means that

$$\rho(t)(v) = exp^{2\pi i \lambda(H)} v.$$

Exercise 1. *If the representation is unitary then different weight spaces are orthogonal: $V_\lambda \perp V_{\lambda'}$, $\lambda \neq \lambda'$.*

G has a canonical representation on its Lie algebra , the adjoint representation.

Definition 2. *The roots of G are the weights of the adjoint representation of G on $\mathfrak{g}$. The corresponding weight spaces are called the root spaces, denoted $\mathfrak{g}_\alpha$. The root space for 0 , that is, the trivial representation of T, is $\mathfrak{t}$.*

Thus

$$\mathfrak{g} = t \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha \qquad (*)$$

and by the lemma, and the bi-invariance of the metric

$$\mathfrak{t}^\perp = \bigoplus \mathfrak{g}_\alpha.$$

Remark on Reality. The Lie algebra of compact G is a real vector space, not a complex one, so its adjoint representation is a *real* representation, not a complex one. But in our discussion of weights above we dealt with complex representations. Our realization (*) is tantamount to identifying each non-trivial real representation of T occuring in the adjoint representation with a fixed complex one-dimensional representation. This in turn is the same as choosing an orientation on the real irreducible representation space.

The complex representation $V_\lambda \cong \mathbb{C}$ of a torus T can be viewed as a real representation, by viewing $\mathbb{C}$ as a real vector space. Then its matrix realization is $exp(H) \mapsto \begin{pmatrix} \cos(2\pi\lambda(H)) & -\sin(2\pi\lambda(H)) \\ \sin(2\pi\lambda(H)) & \cos(2\pi\lambda(H)) \end{pmatrix}$. Conversely, all non-trivial real representations of T are of this form. For if V is a real irrep, then $V \otimes \mathbb{C}$ is a complex rep. which can then be broken into irreducibles: $V_\mathbb{C} = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \ldots$. Because the trace (character) of a real rep is a real function, we must have that the λ_i come in positive-negative pairs: $\lambda_2 = -\lambda_1, \ldots$. Irreducibilty shows us that there are just two summands $V_\lambda \oplus V_{-\lambda}$. A choice of orientation of V is equivalent to identifying V with one of V_λ or $V_{-\lambda}$. Reversing orientation, reverses the choice. Viewed as real representations, and not fixing orientation, we have $V_\lambda \cong V_{-\lambda}$. Summarizing, over the reals, the irreducible representations of T are parametrized by $\hat{T}/\pm \cong \mathbb{Z}^n/\pm 1$ where 0 corresponds to the trivial one-dimensional real representation, and all other representations are the real two-dimensional representations, $V_\lambda \cong V_{-\lambda}$.

The root space for 0 is $\mathfrak{t}$. Because T is Abelian, its own adjoint action is trivial, which shows that $\mathfrak{t}$ is contained in the root space for 0: $Ad_t(v) = v$ for $v \in \mathfrak{t}$. There is something to prove though , hidden in the definition of root space and this is that $\mathfrak{t}$ exhausts the vectors of weight 0. For, suppose there is some other vector $v \in \mathfrak{g}$, $v \notin \mathfrak{t}$, v of weight zero under the adjoint action restricted to T. Then $Ad_t(v) = v$

for all $t \in T$. Exponentiating, $t(exp(sv))t^{-1} = exp(sv)$ for all s. This shows that adjoining the one-parameter subgroup generated by v to T (and taking closures if necessary) yields a compact Abelian group strictly bigger than T, contradicting T's maximality.

Lemma 1. *Under the bi-invariant metric, this decomposition is orthogonal, with* $\mathfrak{t}$ *the tangent space to* T*. For generic* $x \in T$*, the orthogonal sum* $\bigoplus \mathfrak{g}_\alpha$ *is the the tangent space to the conjugacy class* $G \cdot x$ *through* x*.*

If λ is a root, $t = \exp(H) \in T$ and $X \in V_\lambda$ then this means that

$$Ad_t(X) = exp^{2\pi i \lambda(H)} X.$$

Now In a little bit, we will see that all maximal torii are conjugate. If V a complex representation of G. Then, by differentiation, V becomes a representation of the Lie algebra $\mathfrak{g}$, and of the complex Lie algebra $\mathfrak{g}_C$. with Cartan subalgebra $\mathfrak{t}$. A *weight* for V is an element $w \in \mathfrak{t}^*$ such that there is a nonzero vector $v \in V$ with the property that $\zeta \cdot v = w(\zeta)v$ for all $\zeta \in \mathfrak{t}$ (a simultaneous eigenvector). The space of v's for a given weight w is called the *weight space* for w and is denoted V_w. If $w \in \mathfrak{t}^*$ is not a weight we set $V_w = 0$. For a finite-dimensional representation V of $\mathfrak{g}$ the set of weights is finite, and

$$V = \bigoplus_{w \in \mathfrak{t}^*} V_w.$$

The roots of $\mathfrak{g}$ are the weights of the adjoint representation, with the corresponding weight spaces called the root spaces, and denoted by $\mathfrak{g}_\alpha$.

From $\zeta\xi v = \xi\zeta v + [\zeta, \xi]v$ it follows that if $v \in V_w$ and $\xi \in \mathfrak{g}_\alpha$ then $\xi \cdot v \in V_{w+\alpha}$. In other words, $\mathfrak{g}_\alpha \cdot V_w \subset V_{w+\alpha}$, which implies the following "vanishing weight criterion":

> If w is weight and α is a root such that $w + \alpha$ is *not* a weight then $\mathfrak{g}_\alpha \cdot V_w = 0$.

This is part of the proposition

$$\mathfrak{g}_\alpha \cdot V_w \neq 0 \Longleftrightarrow w + \alpha \text{ is a weight.} \tag{1}$$

It follows that if, as in our case, all weight spaces are 1-dimensional, then $\mathfrak{g}_\alpha \cdot V_w = V_{w+\alpha}$ whenever $w + \alpha$ is a weight.