This note explores the spherical harmonics, and how they form the irreducible representations of $SO(3)$ and how they together in terms of an $sl(2)$ action.

Let P denote the space of complex valued real polynomials on $I\!R^3$, and $P_d \subset P$ the space of homogeneous degree d polynomials. Thus

$$P = \bigoplus_d P_d.$$

Exercise 1. Compute the dimension of P_d. Prove that $dim(P_d) = dim(P_{d-2}) + (2d+1)$.

The group $SO(3)$ of rotations of 3-space acts on the space of all continuous complex valued functions on $I\!R^3$ according to $f \mapsto Rf := f \circ R^{-1}$, $R \in SO(3)$. This action maps P to P and perserves degree so defines a finite-dimensional representation of $SO(3)$ on P_d. The Laplacian $\Delta = \frac{\partial}{\partial x}^2 + \frac{\partial}{\partial y}^2 + \frac{\partial}{\partial y}^2$ is a rotationally invariant operator: $\delta(R \cdot f) = R \cdot \Delta f$. It follows that

$$\mathcal{H}_d = P_d \cap ker(\Delta)$$

is an $SO(3)$- invariant subspace.

Definition 1 *$\mathcal{H}_d = P_d \cap ker(\Delta)$ is called the space of degree d harmonic functions. A function f is called "harmonic" if $\Delta f = 0$.*

Example: $(x+iy)^5 \in \mathcal{H}_5 \subset P_5$.

Theorem 1
- *A) $\mathcal{H}_d$ forms an irrep for $SO(3)$, isomorphic to the spin $2d$ irred.*
- *B) P_d decomposes into $SO(3)$-irreps according to :*

$$P_d = \mathcal{H}_d \oplus r^2\mathcal{H}_{d-2} \oplus r^4\mathcal{H}_{d-4} \oplus \dots$$

Exercise. For $d = 4$ show that upon taking dimensions of both sides this decomposition yields the equality $15 = 9 + 5 + 1$.

By the Bolzano-Weierstrass theorem, the restiction of functions $f \in P$ to the unit sphere $S^2 \subset I\!R^3$ forms a dense subset of $C^0(S^2, \mathbf{C})$ or of $L^2(S^2)$. Since rotations map S^2 to itself, they define a representation on these (infinite-dimensional) vector spaces.

Corollary 1 *The space $\mathcal{H}_d|_{S^2}$ is an irrep for $SO(3)$ and is the eigenspace for the Laplacian on the sphere with eigenvalue $d(d+1)$. These spaces give the direct sum decompostion of $L^2(S^2)$ into eigenspaces for the Laplacian:*

$$L^2(S^2) = \bigoplus \mathcal{H}_d|_{S^2}, \qquad \Delta\psi_d = d(d+1)\psi_d, \psi_d \in \mathcal{H}_d|_{S^2}.$$

The $2d+1$ dimensional subspace $\mathcal{H}_d|_{S^2} \subset C^0(S^2, \mathbf{C})$ is called the space of spherical harmonics of degree d.

Proof of theorem. Both the real and imaginary parts of any holomorphic function on the xy plane is harmonic on the plane $\mathbb{R}^2$. We can view such a function as a function on $\mathbb{R}^3$ which is independent of the third z variable. Thus for each positive integer d we have that $\psi_d \in \mathcal{H}_d$ where $\psi_d(x, y, z) = (x + iy)^d$. But ψ_d transforms by $e^{id\theta}$ under rotations of the xy plane by θ so has weight d. It follows that the invariant subspace $\mathcal{H}_d$ contains a copy of the $2d+1$ irrep V_d within it. We will show that $\mathcal{H}_d = V_d$ be a dimension count.

It is enough to show that Δ is onto. For then, since $ker(\Delta) = \mathcal{H}_d$ we would have $dim(P_d) = dim(\mathcal{H}_d) + dim(P_{d-2})$. But by the exercise 1 above, $dim(P_d) = 2d + 1 + dim(P_{d-2})$, so this would prove that $\mathcal{H}_d = V_d$. Inspired by Schur's lemma, we expect that Δ is a "multiple of the identity" on each factor $r^{2s}\mathcal{H}_{d-2s}$. More precisely, both the operators m_{r^2} of multiplication by r^2 and Δ. are $SO(3)$-equivariant. One raises degree by 2 and one lowers degree by 2, so the composition $\Delta \circ m_{r^2}$ maps P_{d-2} to itself, and its image is contained in $\Delta(P_d)$ since m_{r^2} is in

$$m_{r^2} : P_{d-2} \to P_d, \qquad \Delta : P_d \to P_{d-2}.$$

We will show that $\Delta \circ m_{r^2}$, upon restriction to $r^{2(s-2)}\mathcal{H}_{d-2s-2}$ is a positive scalar multiple of the identity. Combined with induction, this will complete the proof.

We now show by hand that the operator $\Delta \circ m_{r^2}$ is a positive scalar multiple of the identity when restricted to each factor $r^{2s}\mathcal{H}_j$. Start with $s = 1$. From $\Delta(fg) = (\Delta f)g + 2\nabla f \cdot \nabla g + f\Delta g$ we compute that $\Delta(r^2 g) = 6g + 4E[g] + r^2\Delta g$ where $E[g] = (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial y})(g)$. E is called the "Euler vector field". Euler's identity asserts that if $E \in P_j$ then $E[g] = jg$. Thus if $g \in \mathcal{H}_{d-2}$ we have that $\Delta(r^2 g) = (6 + 4(d-2))g$.

Exercise. If $\psi \in \mathcal{H}_j$ show that $\Delta(r^{2s}\psi_j) = [s((s-1) + 6 + 4sj]r^{2(s-1)}\psi_j$.

According to the exercise then $(\Delta \circ m_{r^2})(r^{2(s-1)}\psi_{d-2s-2}) = [s((s-1) + 6 + 4s(d-2s-2)](r^{2(s-1)}\psi_{d-2s-2})$ Since the scalar $[s((s-1)+6+4s(d-2s-2)] > 0$ for $0 < s < (d-1)/2$ we have proved that Δ is indeed onto, *provided* we know the decomposition (B) of the theorem is valid for P_{d-2}.

To finish off we proceed by induction. For $d = 0$ and 1, the decomposition is immediately true, with only one factor. Make the inductive hypothesis: that the decomposition (B) holds for P_s for all $s < d$. We show that the decomposition holds for $s = d$. By the inductive hypothesis then, we have the desired decomposition for $s = d-2$ for which to apply our above reasoning to $\Delta : P_d \to P_{d-2}$. Thus, Δ is onto, and so $P_d = V_d \oplus r^2 P_{d-2}$ and $V_d = \mathcal{H}_d$.

QED

Proof of Corollary.

Since $r = 1$ on the sphere, the restrictions of $\mathcal{H}_d$ and of $r^{2m}\mathcal{H}_d$ to the sphere is the same finite dimensional function space. Thus the restriction of P to the sphere is the same as the restriction of the direct sum of the $\mathcal{H}_d$ to the sphere. This proves that this direct sum is dense in $L^2(S^2)$. Now when expressed in spherical coordinates r, θ, ϕ, the $\mathbb{R}^3$ the Laplacian becomes:

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2}\Delta_{S^2}$$

where

$$\Delta_{S^2} f = \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial}{\partial\phi} f) + \frac{1}{\sin^2\phi}(\frac{\partial}{\partial\theta}(\frac{\partial}{\partial\theta} f))$$

for f a function of r, θ, ϕ. Now this latter operator is the Laplacian on the unit sphere, once we set $r = 1$.

Now if $\psi \in P_d$ then $\psi = r^d \beta(\theta, \phi)$ from which we compute that $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r})\psi = (d+1)dr^{d-2}\beta(\theta, \phi)$, which is to say that $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r})\psi = d(d+1)\psi$ upon restriction, (after taking derivatives!) to $r = 1$. Hence for $\psi_d \in \mathcal{H}_d$ we get

$$0 = d(d+1)\psi_d + \Delta_{S^2}\psi_d,$$

upon setting $r = 1$.

Now, use the fact that 'analyst's laplacian, is the negative of the 'geometer's Laplacian' to obtain the corollary.

An $sl(2)$ in the picture! Extension to general dimension.

In our computation we used multiplication by r^2 in combination with the Laplacian Δ. The multiplication operator increased degree by 2 while the Laplacian lowers degree by 2. Hence the commutator $[\Delta, r^2]$ preserves degree.

Proposition 1 *Restricted to P_d, the commutator $[\Delta, r^2]$ is $6 + 4d$ times the identity.*

This proposition is a special case of a result valid in n dimensions. Let $E = \Sigma x_i \frac{\partial}{\partial x^i}$ be the "Euler operator" - the infinitesimal generator of dilations. Write $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ for the transformartion of scalar multiplication by λ: $\delta_\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$ Then $E[f] = \frac{d}{d\lambda}|_{\lambda=1}\delta^*_\lambda f$. Euler's identity asserts that $Ef = df$ for $f \in P_d$, where now P_d denotes the space of homogeneous degree d polynomials on $\mathbb{R}^n$. Also $\nabla r^2 = 2E$ and $\Delta r^2 = 2n$ where $r^2 = \Sigma x_i^2$ is the squared distance from the origin in $\mathbb{R}^n$. It follows from $\Delta(fg) = (\Delta f)g + 2\nabla f \cdot \nabla g + f\Delta g$, with $f = r^2$ that $\Delta(r^2\psi) = 2n\psi + 4E[\psi] + r^2\Delta\psi$, or

$$[\Delta, r^2] = 2nId. + 4E$$

It follows from the degree 2 homogeneity of r^2 that m_{r^2} is homogeneous of degree 2: $\delta_\lambda \circ m_{r^2} \circ \delta_{1/\lambda} = \lambda^2 m_{r^2}$. Setting $\lambda = e^t$ and differentiating this identity at $t = 0$ yields $[E, m_{r^2}] = 2m_{r^2}$. In a similar manner Δ is an operator which is homogeneous of degree -2 and so $[E, \Delta] = -2\Delta$. It follows from these commutation relations that$H = 2nId. + 4E$, $X = m_{r^2}$ and $Y = \Delta$ define a representation of the Lie algebra $sl(2, \mathbb{R})$ on the space P of polynomials on $\mathbb{R}^n$.

Following Hermann Weyl, Roger Howe, combined this $sl(2)$ action with the $O(n)$ action on P and showed the two actions form what is now called a "Howe dual pair". As a consequence of their 'duality' Howe obtains a proof of Theorem 1 for the case of general n. In that theorem $\mathbb{R}^n$ replaces $\mathbb{R}^3$, $SO(n)$ (or $O(n)$ replaces $SO(3)$, and the Laplacian becomes the usual $\mathbb{R}^n$-Laplacian. The corollary also holds, with the sphere now the standard unit $n - 1$ sphere with its induced metric.