

## SLICES OF GROUP ACTIONS.

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By a ‘G-space’ we mean a smooth manifold  $X$  on which a Lie group  $G$  acts smoothly. Associated to each point  $x_0$  of a  $G$ -space  $X$  we have two objects, the *orbit* through  $x_0$ , denoted  $Gx_0$ , which is a subset of  $X$ , and the *isotropy* of  $x_0$  (also called its stabilizer, symmetry group, or little group), denoted  $G_{x_0}$  and which is a closed subgroup of  $G$ . In set-theory notation :

$$Gx_0 = \{y \in X : \exists g \in G, y = gx_0\}, G_{x_0} = \{g \in G : gx_0 = x_0\}.$$

The orbit is a submanifold of  $X$ , and is compact and embedded if  $G$  is compact. The isotropy group is a Lie subgroup of  $G$ . The smooth map

$$\pi_{x_0} : G \rightarrow Gx_0; g \mapsto gx_0$$

relates the orbit and isotropy. The map is constant under the RIGHT  $G_{x_0}$  action on  $G$ , that is  $\pi_{x_0}(gh) = \pi_{x_0}(g)$  for  $h \in G_{x_0}$ . Thus  $\pi_{x_0}$  induces a  $G$ -equivariant diffeomorphism:

$$G/G_{x_0} \cong Gx_0.$$

Both sides of this isomorphism are *homogeneous spaces*.

**Definition 1.** *A homogeneous space is a  $G$ -space where the  $G$ -action is transitive.*

Thus each orbit of  $X$  is a homogeneous space. All homogeneous spaces  $Y$  are diffeomorphic to some coset space  $G/H$  for some closed subspace  $H \subset G$ ,  $H$  being the isotropy of a fixed point of  $Y$ . To understand the structure of a  $G$ -space  $X$  the strategy is to list all its isotropy groups  $H$ , and then describe how the consequent homogeneous spaces  $G/H$  fit together to form  $X$ .

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**Remark 1.** *The map  $\pi_{x_0}$  gives  $G$  the structure of a smooth principal  $G_{x_0}$  bundle over the orbit. The fibers of this fibration are the cosets  $gG_{x_0}$  and are all diffeomorphic to the subgroup  $G_{x_0}$ . Indeed*

**Exercise 1.** *If  $x$  and  $y$  lie on the same orbit then their isotropy groups are conjugate. Precisely  $G_y = gG_xg^{-1}$  if  $y = gx$ .*

It follows that as  $y$  varies over the orbit  $G/H$  the consequent isotropy groups vary over all conjugates to  $H$ .

**Definition 2.** *By the orbit type of  $x$  we mean the conjugacy class of the stabilizer  $G_x$  of  $x$ . We represent the orbit type by  $G_x$  itself.*

*By the orbit types of a  $G$ -space  $X$  we mean the list of the conjugacy classes of isotropy groups  $G_x$  that occur as  $x$  varies.*

**Example 1.** *Take  $G = SO(3)$  and  $X = \mathbb{R}^3$  with the standard linear action of the rotation group  $SO(3)$  on  $\mathbb{R}^3$ . If  $x_0 \neq 0$  then its isotropy group is the circle  $SO(2)$  of all rotations about the line thru  $x_0$ . The orbit of this  $x_0$  is the sphere of radius  $\|x_0\|$ :*

$$Gx_0 = S^2(\|x_0\|) = SO(3)/SO(2)$$

*The origin  $x_0 = 0$  is itself an orbit whose isotropy is all of  $G$ . There are thus two orbit types, the circle subgroup ( $SO(2)$ ) and the whole group  $G$ . The map*

$$\pi_{x_0} : SO(3) \rightarrow S^2 = Gx_0$$

*is the Hopf fibration, up to a  $\mathbb{Z}_2$ -cover.*

The isotropy group represents the ‘symmetry’ of the point  $x$ . We expect the typical point of  $X$  to be the least symmetric, and hence to have the smallest isotropy group among all possible isotropy groups of points of  $X$ . For example, with  $X = \mathbb{R}^3$  and  $G = SO(3)$  we saw that the typical point had a one-dimensional isotropy group, while one single point (the origin) had the entire three-dimensional group  $G$  as its isotropy group.

We formalize the idea of ‘less symmetric’ with a partial order “ $\succeq$ ” on the collection of closed subgroups of  $G$ . This order measures instead the size of the orbit. Observe that the larger the isotropy group  $H$ , the smaller its corresponding orbit  $G/H$  so that these typical points we expect to have the ‘largest’ isotropy group. Formally, write  $K \succeq H$  if  $K$  lies *inside* some conjugate of  $H$ . In other words:

$$K \succeq H \iff \exists g \in G : K \subset gHg^{-1}.$$

This order is invariant under conjugation and so induces an order on the set of all conjugacy classes of closed subgroups of  $G$ .

**Remark 2.** *Using conjugation, we may assume that  $K \subset H$  when  $K \succeq H$ . We then have an obvious submersion:*

$$(1) \quad K \succeq H \implies \pi : G/K \rightarrow G/H \text{ (onto)}$$

*between homogeneous spaces.*

**Exercise 2.** *a) Show the fiber of the fibration (1) is the homogeneous space  $H/K$ .*

*b) Show that there exists a  $G$ -equivariant map  $G/K \rightarrow G/H$  if and only if  $K \succeq H$ .*

**Theorem 1** (Principal orbit type). *If  $G$  is compact and  $X$  is connected then there is a unique maximal orbit type  $H$ , called the principal orbit type for  $X$ . Thus, for all  $x \in X$  there is a  $g \in G$  such that  $gHg^{-1} \subset G_x$ . The set of points with orbit type  $H$  forms a dense open connected subset of  $X$ .*

In the example of  $SO(3)$  on  $\mathbb{R}^3$  the principal orbit type is  $SO(2)$ .

To prove the maximal torus theorem, we think of  $G$  itself as an  $X$ . We will see that the maximal torus theorem is almost equivalent to the following theorem

**Theorem 2** (Principal orbit type for conjugation action). *Let  $G$  be a compact connected Lie group. Consider  $X = G$  to be a  $G$ -space by having  $G$  act on  $G$  by*

conjugation. Then the principal orbit type of this action is realized by (any of) the maximal torus  $T \subset G$ .

The proof will rely on the slice theorem and the isotropy representation so will have to wait for these topics to be presented.

In the case  $G = SO(3)$ , if we differentiate the action of conjugation at the identity we get the adjoint action, which we have seen is the action described in the example above. In other words, the example above can be viewed as the derivative of the previous theorem for the case of  $G = SO(3)$ .

## 1. SLICES

The relation “ $x$  and  $y$  lie on the same orbit” is an equivalence relation on  $X$ , and hence  $X$  is decomposed into the disjoint union of orbits. Each orbit itself is a homogeneous space.  $X$  itself can be recovered by the way these various homogeneous pieces fit together.

This ‘fitting together’ of orbits is typically a complicated mess if  $G$  is non-compact. For example, dynamical systems are just “ $\mathbb{R}$ -spaces” and we see pictures of horribly (or beautifully, depending on your aesthetics!) complicated orbit structures in books on “chaos”. But for a compact Lie group  $G$  the ‘fitting together’ of the orbits is surprisingly well organized. The most effective organizational structure is that of a *slice*.

**Definition 3.** A local slice at  $x_0$  is an embedded disc  $S \rightarrow X$ , transverse to the orbit, invariant under the isotropy group:  $G_{x_0}S = S$  and such that  $GS$  is a neighborhood of the orbit  $Gx_0$ .

A global slice is an embedded submanifold  $S \subset X$  which intersects some orbit  $Gx_0$  transversally, is  $G_{x_0}$  invariant, and intersects every orbit of  $X$ . Thus  $GS = X$ .

From the definition of ‘local slice’  $S$  at  $x$ , we see that if  $y$  is a point close to  $x$  then the  $G$  orbit through  $y$  must intersect the slice  $S$ . This is where the word ‘slice’ comes from: the slice, cuts, or slices across, across all (nearby) orbits. Although

$S$  intersects all nearby orbits, this intersection need not be unique. Understanding the intersection is crucial to understanding how the orbits fit together.

Also note that  $\dim(Gx_0) + \dim(S) = \dim(X)$ , or

$$\dim(G) - \dim(G_{x_0}) + \dim(S) = \dim(X).$$

**Example 2.** *Continuing with  $SO(3)$  acting on  $\mathbb{R}^3$ , for  $x_0 \neq 0$  the line spanned by  $x_0$  is a global slice. Any subinterval of this line which contains  $X_0$  is a ‘local slice’, If the length of that interval is less than  $|x_0|$  then the local slice has the property that it intersects each orbit exactly once.*

**Theorem 3** (Slice theorem). *If  $G$  is a compact Lie group, and  $X$  is a  $G$ -space, then for every point  $x \in X$  there is a slice  $S$  to the  $G$ -action. This slice can be constructed as the ‘orthogonal to the orbit’ at  $x_0$ .*

*If  $X$  is compact or complete, then there is a global slice. It can be obtained by ‘exponentiating’ the normal vectors to the orbit  $Gx_0$  at  $x_0$  with respect to a  $G$ -invariant metric on  $X$ .*

The idea of the proof of the Slice theorem can be seen in the example of  $SO(3)$  acting on  $\mathbb{R}^3$  that we have been carrying along. The line through  $x_0 \neq 0$ , which we saw is a slice, is precisely the line orthogonal to the orbit through  $x_0$ , which is of course the sphere of radius  $\|x_0\|$ .

**Theorem 4** (Slice theorem, case of an orthogonal representation). *Let  $\mathbb{E}$  be an orthogonal representation space for  $G$ . View  $\mathbb{E}$  as a  $G$ -space. Then the tangent space to the orbit through  $v \in \mathbb{E}$  is  $T_v Gv = \mathfrak{g} \cdot v$ , and a global slice to the  $G$ -action at  $v$  is given by taking the orthogonal complement to this tangent space, namely, the slice is  $S = (\mathfrak{g} \cdot v)^\perp$ .*

Proof. The representation is a map  $G \times \mathbb{E} \rightarrow \mathbb{E}$  which we will write as  $(g, v) \mapsto g \cdot v$  and which is linear in  $v$ , for each fixed  $g$ . Differentiating with respect to  $g$  at  $g = e$  we get the ‘infinitesimal’, or Lie algebra action  $\mathfrak{g} \times \mathbb{E} \rightarrow \mathbb{E}$ , expressed  $\xi, v \mapsto \xi \cdot v$  which is a bilinear map. If we fix  $v$  and vary  $g$  we sweep out the orbit  $Gv$ , which

shows that the tangent space  $T_v(Gv)$  consists of all vectors of the form  $\xi \mapsto \xi \cdot v$  as  $\xi$  varies over  $\mathfrak{g}$ . This subspace we have written as  $\mathfrak{g} \cdot v$ .

Since the representation is orthogonal  $Gv$  is contained in the sphere  $S^{N-1}(\|v\|)$  of radius  $\|v\|$  about the origin. Consequently  $T_v(Gv) \subset T_v S^{N-1}(\|v\|) = v^\perp$ . Taking perps of this inclusion we get  $v \in T_v(Gv)^\perp$  so that indeed  $v \in S$  as is required.  $S$  intersects  $T_v(Gv)$  orthogonally by construction.

It remains to show that  $S$  intersects every  $G$  orbit. To this purpose, let  $w$  be any point of  $\mathbb{E}$  and consider its orbit  $Gw$ .  $Gw$  and  $Gv$  are smooth compact submanifolds of  $\mathbb{E}$ , disjoint from each other. Consider the infimum of all the distances  $\|x - y\|$  as  $x$  varies over  $Gv$  and  $y$  varies over  $Gw$ . This infimum is realized and the resulting minimizing pair  $(A, B)$ ,  $A \in Gv, B \in Gw$  forms a line segment  $AB$  which must be orthogonal to each orbit at its respective endpoint. Now  $A = gv$  for some  $g \in G$ . Apply  $g^{-1}$  to rotate the line segment to  $vB', B' = g^{-1}B$ . We have that  $B' \in Gw$ , and since the  $G$  action is orthogonal, the rotated line segment remains orthogonal to both orbits. (Its length also equals  $\|A - B\|$ .) In particular  $B' - v \in S$ . Since  $v \in S$  and  $S$  is a linear subspace we have that  $B' = v + (B' - v) \in S$ , proving that  $Gw \cap S \neq \emptyset$

QED

NOW DO?? ADJOINT Action of  $G$  on  $\mathfrak{g}$ , compact case??

this is the ‘infinitesimal maximal torus theorem’ !

## 2. INVARIANT METRICS

We collect the tools needed to prove the slice theorem: the tubular neighborhood theorem, existence of invariant metrics, the equivariant tubular neighborhood theorem, and the isotropy representation.

From now on, let  $G$  be compact. Use the fact that Haar measure is bi-invariant. Then, starting with any metric on  $X$  (i.e Riemannian metric), by pushing it forward by the  $G$ -action and averaging the result with respect to Haar measure, we obtain a Riemannian metric on  $X$  such that this action of  $G$  on  $X$  is an action by isometries.

Now recall how we use the Riemannian exponential map to construct a ‘tubular neighborhood’ of any embedded submanifold  $\Sigma \subset X$ . (In a moment  $\Sigma$  will be  $Gx_0$ .)

#### THE RIEMANNIAN EXPONENTIAL MAP.

First, to define the exponential map, we use the fact that the geodesic equation is a second-order differential equation on  $X$  (a first order equation on  $TX$ ) and as a result, there is a unique geodesic  $\gamma(t)$  passing through any point  $x$ , and pointing in any direction  $v \in T_x X$ . Here, ‘passing through  $x$ ’ means that  $\gamma(0) = x$ , and ‘pointing in the direction  $v$ ’ means that  $\dot{\gamma}(0) = v$ . Then the Riemannian exponential map is the map  $exp_{R,x} : T_x X \rightarrow X$  given by  $exp_{R,x}(v) = \gamma(1)$ . It satisfies the property that  $exp_{R,x}(tv) = \gamma(t)$ .

#### THE TWO EXPONENTIAL MAPS AGREE!

**Theorem 5.** *For a bi-invariant metric on a Lie group, the Riemannian exponential with  $x = 1$  agrees with the group theoretic exponential.*

We proved a special case of this theorem, that of  $G = U(n)$ , in class today.

#### TUBULAR NEIGHBORHOODS

Return to our general submanifold  $\Sigma \subset X$ . The ‘normal bundle’ to  $\Sigma$  is the vector bundle over  $\Sigma$  whose fiber over  $x$  consists of all the vectors  $v \in T_x X$  such that  $v \perp T_x \Sigma$ . We write it  $\nu(\Sigma)$ . Thus

$$\nu(\Sigma)_x = (T_x \Sigma)^\perp.$$

Then the normal exponential map is the smooth map

$$exp_\Sigma : \nu(\Sigma) \rightarrow X$$

defined by  $exp_\Sigma(x, v) = exp_{R,x}(v)$ . Note that the curves  $exp_\Sigma(x, tv)$  are geodesics passing through  $x \in \Sigma$  and orthogonal to  $\Sigma$ .

**Theorem 6** (Tubular neighborhood theorem). *The normal exponential map is a diffeomorphism in a neighborhood of the zero section, thus making a neighborhood*

of  $\Sigma$  in  $X$  diffeomorphic to a disc bundle lying inside the vector bundle  $\nu(\Sigma)$ . This diffeo sends the zero section to  $\Sigma$ .

The slice theorem is essentially an equivariant tubular neighborhood theorem. We apply the construction of the theorem with care, with  $\Sigma = Gx_0$  and the metric on  $X$  being  $G$ -invariant.

### 3. ISOTROPY REPRESENTATION

For  $h \in G_{x_0}$  we have that  $h(x_0) = x_0$ . Differentiating, we see that  $(dh)_{x_0} : T_{x_0}X \rightarrow T_{x_0}X$  is a linear map, where, by slight abuse of notation we have written  $dh$  for the differential of map  $X \rightarrow X$  given by  $x \mapsto hx$ . By the chain rule,  $d(hg)_{x_0} = dh_{x_0}dg_{x_0}$  if  $h, g \in G_{x_0}$  so this differentiation defines a linear representation of the isotropy group on the tangent space of the manifold.

**Definition 4.** *The isotropy representation at  $x_0 \in X$  is the linear representation of the isotropy group  $G_{x_0}$  of  $x_0$  on the tangent space  $T_{x_0}X$  at  $x_0$  obtained by differentiation as above.*

For  $G$  compact the isotropy representation splits as

$$T_{x_0}X = T_{x_0}(Gx_0) \oplus \nu_{x_0}; \text{ with } \nu_{x_0} = T_{x_0}(Gx_0)^\perp.$$

the  $\perp$  taken with respect to the invariant Riemannian metric, which is a  $G_{x_0}$  invariant inner product on  $T_{x_0}X$ .

The first part of the slice theorem follows immediately from:

**Proposition 1.** *A local slice  $S$  at  $x_0$  is given*

$$S = \exp_{x_0}(D); D = D(\delta) \subset \nu_{x_0}$$

*with  $D$  being a small disc centered at the origin.*

To establish the proposition, observe that  $G$  acts on the normal bundle by differentiation, and this action covers the action of  $G$  on the orbit. To be precise, the  $G$



action on the normal bundle is given by:  $g(x, v) = (gx, dg_x(v))$  for  $v \in \nu_x, x \in Gx_0$ . (This action turns  $\nu(Gx_0)$  into what is called a “homogeneous vector bundle”.) Because  $G$  acts by isometries, the normal exponential map becomes a  $G$ -equivariant map:

$$\exp_{Gx_0} : \nu(Gx_0) \rightarrow X.$$

(“Equivariant” means  $\exp(g(x, v)) = g \exp(x, v)$  for  $x \in Gx_0, v \in \nu_x$ .) The normal exponential map takes a disc bundle of the zero section diffeomorphically onto a neighborhood of  $Gx_0$ . Each fiber of this disc bundle has the form  $g(D)$ , where  $D \subset \nu_{x_0}$  is as in the proposition, so that equivariance implies that  $gS$  sweeps out a neighborhood of the orbit. Again, equivariance implies that each  $h \in G_{x_0}$  maps  $S$  to itself.

To finish the proof of the slice theorem, we need the global slice  $S$ . Simply replace  $D$  by all of  $\nu_{x_0}$  so that  $S = \exp(\nu_{x_0})$ . That  $S$  intersects every orbit essentially follows from the fact that the exponential map at  $x_0$  maps  $T_{x_0}X$  onto  $X$  when  $X$  is compact or complete.

#### 4. PRINCIPAL ORBIT TYPE VIA ISOTROPY REPRESENTATION.

The following propositions are immediate consequences of the slice theorem.

**Proposition 2** (Upper semicontinuity of orbit type). *For all  $x \in X$  there exists a nbhd  $U$  of  $x$  such that for all  $y \in U$ ,  $G_y \succeq G_x$ .*

Proof. By the slice theorem, we can assume that  $y \in S$ , the local slice at

**Proposition 3.** *The orbit type of a point  $x \in X$  is locally constant if and only if the isotropy representation on the space  $\nu_x$  of normal vectors at  $x$  is the trivial representation.*

It suffices to show that for all  $y \in S$ ,  $y$  sufficiently close to  $x$ , that  $G_y$

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This means that  $G$  acts on all of  $\nu$  by vector bundle automorphisms, and this action covers the  $G$  action on the orbit:

$$dg_x : \nu(G_{x_0}) \rightarrow \nu(G_{x_0}); g \in G.$$

Recall that a representation of  $G$  on  $V$  is called “irreducible” if the only  $G$ -invariant linear subspaces of  $V$  are  $V$  itself and the 0 subspace. If the orbit through  $x_0$  is not open, then the isotropy representation is not irreducible, since the tangent space to the orbit is an invariant subspace.

We will see momentarily that if  $G$  is compact then the isotropy representation always splits:

The action of the isotropy group on the normal piece  $\nu_{x_0}$  is what tells us how to “piece together” orbits.

ISOTROPY REPRESENTATION Since the isotropy group  $H = G_{x_0}$  leaves  $x_0$  invariant, we have that  $H$  acts linearly on  $T_{x_0}X$ . This action is called the “isotropy representation”. Now in our case, of  $G$  compact acting on  $X$  by isometries, this representation splits into two sub-representations, tangential and normal:

$$T_{x_0}X = T_{x_0}(Gx_0) \oplus \nu_x(Gx_0)$$

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