Homework #4

+2 for except. concision → 12/10

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A:

$$m_1\ddot{\vec{q_1}} = -\frac{\delta}{\delta\vec{q_1}}V_\alpha = G(\alpha)\frac{\delta}{\delta\vec{q_1}}\left[\frac{m_1m_2}{(r_{12})^\alpha} + \frac{m_2m_3}{(r_{23})^\alpha} + \frac{m_1m_3}{(r_{13})^\alpha}\right]$$

Note that $\frac{\delta}{\delta\vec{q_1}}\frac{m_2m_3}{(r_{23})^\alpha} = 0$, because $r_{23} = \|q_2 - q_3\|$ does not depend on q_1.
Similarly, noting that $\frac{\delta}{\delta\vec{q_1}}r_{12} = \frac{\delta}{\delta\vec{q_1}}\|q_1 - q_2\| = \frac{q_1-q_2}{\|q_1-q_2\|} = \frac{q_1-q_2}{r_{12}}$, and likewise $\frac{\delta}{\delta\vec{q_1}}r_{13} = \frac{q_1-q_3}{r_{13}}$, we continue our evaluation via the chain rule:

$$m_1\ddot{\vec{q_1}} = G(\alpha)\left[-\alpha\frac{m_1m_2}{(r_{12})^{\alpha+1}}\left(\frac{\delta}{\delta\vec{q_1}}r_{12}\right) + 0 - \alpha\frac{m_1m_3}{(r_{13})^{\alpha+1}}\left(\frac{\delta}{\delta\vec{q_1}}r_{13}\right)\right]$$

$$= -\alpha G(\alpha)\left[\frac{m_1m_2(q_1-q_2)}{(r_{12})^{\alpha+2}} + \frac{m_1m_3(q_1-q_3)}{(r_{13})^{\alpha+2}}\right]$$

Thus $F_{12} = -\alpha G(\alpha)\frac{m_1m_2(q_1-q_2)}{(r_{12})^{\alpha+2}} = G(\alpha)\alpha m_1m_2\frac{q_2-q_1}{(r_{12})^{\alpha+2}}$, and it follows similarly that

$$F_{ab} = G(\alpha)\alpha m_am_b\frac{q_b-q_a}{\|q_b-q_a\|^{\alpha+2}}$$

✓

B:

C: To show that $E = K + V_\alpha$ is constant, we evaluate $\frac{dE}{dt} = \frac{d}{dt}\sum\frac{1}{2}m_iq_i{}^2 + \frac{d}{dt}V_\alpha = \frac{d}{dt}\frac{1}{2}\left\langle\dot{\vec{q}},\dot{\vec{q}}\right\rangle - \frac{d}{dt}U_\alpha = \left\langle\dot{\vec{q}},\ddot{\vec{q}}\right\rangle - \frac{d}{dt}U_\alpha(q) = \left\langle\dot{\vec{q}},\ddot{\vec{q}}\right\rangle - \langle\nabla U_\alpha(q),\dot{q}\rangle = \left\langle\dot{\vec{q}},\ddot{\vec{q}} - \nabla U_\alpha(q)\right\rangle = \left\langle\dot{\vec{q}},0\right\rangle = 0$ ✓

D: $I(q) = \langle q,q\rangle$, so $\frac{d^2I}{dt^2} = 2\langle\dot{q},\dot{q}\rangle + 2\langle q,\ddot{q}\rangle = 4K + 2\langle q,\nabla U_\alpha\rangle$. We now use the fact that U_α is homogeneous of degree $-\alpha$ (since all the terms are of the form $\frac{1}{r^\alpha}$) to show that $\langle\nabla U_\alpha(q),q\rangle = -\alpha U_\alpha(q)$. From here, we have $\frac{d^2I}{dt^2} = 4K + 2\langle q,\nabla U_\alpha\rangle = 4K - 2\alpha U_\alpha$ ✓

E: When $\alpha = 2$, then $\frac{d^2I}{dt^2} = 4K - 4U_2 = 4(K - U_2) = 4E$ Antidifferentiating twice gives us $\frac{dI}{dt} = 4Et + \dot{I}(0)$ and $I = 2Et^2 + \dot{I}(0)t + I(0)$

✓

you are too slick! *