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HW4

$\frac{12}{10}$ ← extra 2 for exceptional care & clarity of exposition!

(A) We know $m_a \ddot{q}_a = F_{ab} + F_{ac}$ also from class $m_a \ddot{q}_a = \frac{\partial U}{\partial q_a}$ where $U = -V$ so if we replace V by $V_\alpha = -U_\alpha$ we have $m_a \ddot{q}_a = \frac{\partial U_\alpha}{\partial q_a}$ so to find F_{ab} and F_{ac} we can find the partial derivative of U_α with respect to q_a. Here we will do it for $a=1$, but the case is analogous for $a=2$ and $a=3$.

$$\frac{\partial}{\partial q_1}\left(\underset{\substack{\uparrow\\ \text{constant}}}{G(\alpha)}\left(\frac{m_1 m_2}{r_{12}^\alpha} + \underset{\substack{\uparrow\\ \text{treated as constant}}}{\frac{m_2 m_3}{r_{23}^\alpha}} + \frac{m_1 m_3}{r_{13}^\alpha}\right)\right)$$

$$= G(\alpha)\left(\frac{\partial}{\partial q_1}\left(\frac{m_1 m_2}{r_{12}^\alpha}\right) + \frac{\partial}{\partial q_1}\left(\frac{m_1 m_3}{r_{13}^\alpha}\right)\right)$$

we will focus on this term, the other is analogous.

$$m_1 m_2 \frac{\partial}{\partial q_1} r_{12}^{-\alpha} \underset{\substack{\uparrow\\ \text{chain rule}}}{=} m_1 m_2 \left(-\alpha\, r_{12}^{-\alpha-1} \cdot \frac{\partial}{\partial q_1} r_{12}\right)$$

$r_{12} = |q_1 - q_2| = \sqrt{x_{12}^2 + y_{12}^2 + z_{12}^2}$ where $x_{12} = (x_1 - x_2)$ etc. and $q_1 = (x_1, y_1, z_1)$ etc.

$$\frac{\partial (x_{12}^2 + y_{12}^2 + z_{12}^2)^{1/2}}{\partial q_1} \underset{\substack{\uparrow\\ \text{chain rule}}}{=} \frac{1}{2\sqrt{x_{12}^2 + y_{12}^2 + z_{12}^2}} \cdot \frac{\partial}{\partial q}(\cdot)$$

$(\cdot) = x_{12}^2 + y_{12}^2 + z_{12}^2$

$$\frac{\partial ((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)}{\partial (x_1, y_1, z_1)} = 2(x_{12}, y_{12}, z_{12}) = 2(q_1 - q_2)$$

y's and z's constant, x_2 constant, chain rule we get $2x_{12}$
similarly for y_1 we get $2y_{12}$ and for z_1 we get $2z_{12}$

continues

so $\frac{\partial r_{12}}{\partial q_1} = \frac{\cancel{2}(q_1 - q_2)}{\cancel{2}|q_1 - q_2|} = \frac{(q_1 - q_2)}{r_{12}}$

and

$$\frac{\partial}{\partial q_1}\left(\frac{m_1 m_2}{r_{12}^{\alpha}}\right) = m_1 m_2 \Big(\underbrace{-\alpha r_{12}^{-(\alpha+1)}}_{\frac{-\alpha}{r_{12}^{\alpha+1}}} \cdot \underbrace{\frac{(q_1 - q_2)}{r_{12}}}_{\frac{(q_1-q_2)}{r_{12}}} \Big) = \frac{-m_1 m_2 \alpha (q_1 - q_2)}{r_{12}^{\alpha+2}}$$

$$\underbrace{\frac{-\alpha}{r_{12}^{\alpha+1}} \cdot \frac{(q_1-q_2)}{r_{12}}}_{\frac{-\alpha(q_1-q_2)}{r_{12}^{\alpha+2}}}$$

and

$$\frac{\partial U_\alpha}{\partial q_1} = -G(\alpha)\left(\frac{m_1 m_2 \alpha (q_1 - q_2)}{r_{12}^{\alpha+2}} + \frac{m_1 m_3 \alpha (q_1 - q_3)}{r_{13}^{\alpha+2}}\right)$$

so ~~a~~ in general

$$\frac{\partial U_\alpha}{\partial q_a} = m_a \ddot{q}_a = \overbrace{\underbrace{\frac{-G(\alpha) m_a m_b \alpha (q_a - q_b)}{r_{ab}^{\alpha+2}}}_{F_{ab}}}^{\vec{F}_{ab}} + \overbrace{\underbrace{\frac{-G(\alpha) m_a m_c \alpha (q_a - q_c)}{r_{ac}^{\alpha+2}}}_{F_{ac}}}^{F_{ac}}$$

excellen!

ok!

(B) Argue that $\ddot{q} = \cancel{\frac{\partial U_\alpha(q)}{\partial q}} = \nabla U_\alpha(q)$ where ∇ is with respect to the mass metric as ~~[illegible]~~ in class

From class we know that by definition

$$\nabla U_\alpha(q) = \left(\frac{1}{m_1}\frac{\partial U_\alpha}{\partial q_1}, \frac{1}{m_2}\frac{\partial U_\alpha}{\partial q_2}, \frac{1}{m_3}\frac{\partial U_\alpha}{\partial q_3}\right) \quad (\text{for } N=3)$$

continues

(C) Show that the energy $E = K + V_\alpha$ is conserved that is show $\dot{E} = 0$

$$\dot{K} + \dot{V}_\alpha = 0$$

$$\dot{K} = -\dot{V}_\alpha = \dot{U}_\alpha$$

from class we know $K = \frac{1}{2}\langle \dot{q}, \dot{q} \rangle$ ← weighted by masses

$$\text{so } \dot{K} = \tfrac{1}{2} 2 \langle \dot{q}, \ddot{q} \rangle = \langle \dot{q}, \ddot{q} \rangle$$

from part (B) we know $\ddot{q} = \nabla U_\alpha$ so ← weighted by masses too

$$\dot{K} = \langle \dot{q}, \nabla U_\alpha \rangle$$

Then to get $\dot{U}_\alpha(q)$ we use chain rule: $\left\langle \frac{\partial U_\alpha}{\partial (q)}, \frac{\partial q}{\partial t} \right\rangle$

so $\dot{U}_\alpha = \langle \nabla U_\alpha, \dot{q} \rangle = \langle \dot{q}, \nabla U_\alpha \rangle = \dot{K}$ $\square$

ok!

(D) Let $I = \langle q, q \rangle = m_1 |q_1|^2 + m_2 |q_2|^2 + m_3 |q_3|^2$ be the moment of inertia. Derive the Lagrange-Jacobi identity

$$\ddot{I} = aE + bU_\alpha$$ (i.e. find constants a and b) ↳ in terms of α.

$\dot{I} = 2\langle \dot{q}, q \rangle$

From class we know $K = \frac{1}{2}\langle \dot{q}, \dot{q} \rangle$ and From part (B) we know $\ddot{q} = \nabla U_\alpha$ so

$\ddot{I} = 2(\langle \ddot{q}, q \rangle + \langle \dot{q}, \dot{q} \rangle)$

$= 2(\langle \nabla U_\alpha, q \rangle + 2K)$

good!

U_α is homogeneous degree $-\alpha$ (there is an α as an exponent in each denominator), so we can use Euler's identity for homogeneous fcns

$= 2(-\alpha U_\alpha + 2K) = 4K - 2\alpha U_\alpha$

Let β be such that $2\alpha + \beta = 4$ then add 0

$= 4K \underbrace{-2\alpha U_\alpha - \beta U_\alpha}_{-4U_\alpha} + \beta U_\alpha = 4(K - U_\alpha) + \beta U_\alpha$ (continues ↗)

Ⓓ (continued)

we know
$K - U_\alpha = E$

$\ddot{I} = 4(K - U_\alpha) + \beta U_\alpha$

we set $\beta = 4 - 2\alpha = 2(2-\alpha)$
(s.t. $2\alpha + \beta = 4$)

$$\boxed{\frac{d^2 I}{dt^2} = 4E + 2(2-\alpha)U_\alpha}$$

so the a and b constants we were looking for are $a = 4$
$b = (2-\alpha)2$

Ⓔ Something special happens with the Lagrange-Jacoby identity $\frac{d^2 I}{dt^2} = 4E + 2(2-\alpha)U_\alpha$ when $\alpha = 2$. What is it? Use this to solve for the time evolution $I(t)$ in this case.

$\rightarrow 4E + 2(2-2)U_\alpha = 4E = \ddot{I}$ (the moment of inertia is 4 times the Energy)

4E is a constant so we can take the integrals:

$\ddot{I} = 4E$

$\dot{I} = \int 4E\,dt = 4Et + a$

$I = \int 4Et + a\,dt = \frac{4Et^2}{2} + at + b$

$$\boxed{I(t) = 2Et^2 + at + b}$$

where a and b are constants, $a = \dot{I}$ when time is 0 and $b = I$ when time is 0

excellent!