POLLARD EXERCISES

... problem 2.3 ends:
"between 0 and t_0. First try the case $f(r) = \mu r^{-3}$, then the case $f(r) = \mu r^{-2}$".

EXERCISE 2.1. Set up the equations of motion of a particle moving subject to two distinct centers of attraction, each with its own law of attraction.

EXERCISE 2.2. Suppose that a particle subject to attraction by a fixed center starts from rest, i.e., that at some instant $t = 0$ we have $v = 0$. Then by (2.1) $c = 0$ and the motion is linear. Suppose, moreover, that $f(r)$ is positive for $0 < r < \infty$. Prove that the particle must collide with the center of force in a finite length of time t_0.

EXERCISE 2.3. In the preceding problem, can you tell where the particle will be at each instant of time

FIGURE 1. from section 1. 2; ... problem 2.3 ends: "between 0 and t_0. First try the case $f(r) = \mu r^{-3}$, then the case $f(r) = \mu r^{-2}$.

and is known as the *principle of conservation of energy*.

EXERCISE 3.1. Show that if $f(r) = \mu r^{-p}$, where $p > 1$, then a particle moving with negative energy cannot move indefinitely far from O.

EXERCISE 3.2. Show that if $f(r) = \mu r^{-p}$, then $f_1(r) = \mu(p-1)^{-1} r^{1-p}$ if $p \neq 1$ and $f_1(r) = \mu \log 1/r$ if $p = 1$.

*EXERCISE 3.3. Let $\mathbf{a} = \mathbf{r}$, $\mathbf{b} = \mathbf{v}$ in the standard vector formula

$$(\mathbf{a} \cdot \mathbf{b})^2 + (\mathbf{a} \times \mathbf{b})^2 = a^2 b^2.$$

Conclude that

$$v^2 = \dot{r}^2 + c^2 r^{-2}.$$

What is the physical meaning of the components $\dot{r}$ and $c/\mathbf{r}$ of $\mathbf{v}$? Show that the law of conservation of energy can be written

$$r^2 \dot{r}^2 + c^2 = 2r^2 [f_1(r) + h].$$

FIGURE 2. from section 1.3

*EXERCISE 4.1. Show that if $0 < e < 1$ or $e > 1$ the semi-major axis of the corresponding conic has length a given by the formula

$$\mu a |e^2 - 1| = c^2.$$

EXERCISE 4.2. Use (4.2) to obtain the formula

$$\mu \mathbf{e} = \left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - (\mathbf{r} \cdot \mathbf{v})\mathbf{v}.$$

FIGURE 3. from section 1.3

EXERCISE 5.1. What can you say about the orbit if $f(r) = -\mu r^{-2}$ rather than $f(r) = \mu r^{-2}$? This corresponds to a repulsive force rather than an attraction.

EXERCISE 5.2. Use (5.4) to prove that in the case of elliptical motion the speed of the particle at each position Q is the speed it would acquire in falling to Q from the circumference of a circle with center at O and radius equal to the major axis of the ellipse.

*EXERCISE 5.3. The area of an ellipse is $\pi a^2(1-e^2)^{1/2}$. We already know that if $c \neq 0$ the particle sweeps out area at the rate $c/2$. Combine these facts to show that if $0 < e < 1$ the period p of a particle, that is, the time it takes to sweep out the area once, is given by the formula $p = (2\pi/\sqrt{\mu})a^{3/2}$. This is Kepler's *third law*.

*EXERCISE 5.4. Define the moment of inertia $2I$ by the formula $2I = mr^2$. Write $r^2 = (\mathbf{r} \cdot \mathbf{r})$ and prove that

$$\ddot{I} = 2T - U = T + h_1 = U + 2h_1.$$

In the case of circular motion I is constant so that $2T = U$, a result we already know from Sec. 4.

EXERCISE 5.5. (*Hard.*) Use the preceding exercise to prove that if $c \neq 0$, $h > 0$, then $r/|t|$ approaches $\sqrt{2h}$ as $|t| \to \infty$. (The hypothesis $c \neq 0$ rules out the possibility of a collision with the origin in a finite time.)

FIGURE 4. from section 1.5

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EXERCISE 7.2. Excluding the cases of collision, show that if $h = 0$, then $r|t|^{-2/3} \to (\frac{9}{2}\mu)^{1/3}$ as $|t| \to \infty$. Compare this with the corresponding result in case $h > 0$. (See Ex. 5.5.)

EXERCISE 7.3. Show that $u = (c/\sqrt{\mu})\tan f/2$, thus relating the two anomalies in the case $c \neq 0$. Hint: equate r as given by (7.5) and by (7.6).

FIGURE 5. from section 1.7

EXERCISE 8.1. Show from the Eqs. (8.2) that if $h > 0$, then as $|t| \to \infty$ the ratio $r/|t|$ approaches $\sqrt{2h}$, provided that the value $r = 0$ is not reached at a finite value of t. This gives an alternative solution of Ex. 5.5.

EXERCISE 8.2. Show from the formula $r + \mathbf{e} \cdot \mathbf{r} = c^2/\mu$ that if $h > 0$, $c \neq 0$, the unit vector $\mathbf{r}/r$ approaches a limit vector $\mathbf{l}$ as $t \to \infty$ and that $\mathbf{e} \cdot \mathbf{l} = -1$. Then, according to the formula

$$\mu(\mathbf{c} \times \mathbf{e}) = c^2\mathbf{v} - \mu\,\frac{\mathbf{c} \times \mathbf{r}}{r}\,,$$

easily derived from (4.2), the vector $\mathbf{v}$ also approaches a limit $\mathbf{V}$. What is the length of $\mathbf{V}$?

*EXERCISE 8.3. By matching each of Eqs. (8.2) and (8.3) with (8.5) pair-wise, obtain these formulas connecting true and eccentric anomalies:

$$\tan\frac{f}{2} = \left(\frac{e+1}{e-1}\right)^{1/2}\tanh\frac{u}{2}\,, \qquad h > 0$$

$$\tan\frac{f}{2} = \left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{u}{2}\,, \qquad h < 0.$$

*EXERCISE 8.4. Show that for each value of t each of the equations

$$n(t-T) = e\sinh u - u, \qquad e \geqslant 1$$

$$n(t-T) = u - e\sin u, \qquad 0 < e \leqslant 1$$

has a unique solution u. They are known as *Kepler's equations*.

FIGURE 6. from section 1.8

*EXERCISE 10.2. Let Q_0, Q_1 be two positions on an elliptic orbit, and let u_0, u_1 be the corresponding eccentric anomalies. Assume $u_1 > u_0$. Prove that the distance Q_0Q_1 is $2a \sin \alpha \sin \beta$, where $\alpha = \frac{1}{2}(u_1 - u_0)$ and β is defined by $\cos \beta = e \cos \frac{1}{2}(u_1 + u_0)$, $0 < \beta < \pi$.

EXERCISE 10.3. Prove Lambert's theorem, which says that for an elliptic orbit the time occupied in moving from one position to another depends only on the sum of the distances from O of the two positions, and on the length of the chord joining the positions. (This will be proved in Sec. 11, but try it now, using Ex. 10.2.)

FIGURE 7. from section 1.10

EXERCISE 14.1. The shape of an orbit in the central force problem is determined by the sign of h. Prove from this that in the two-body problem the orbit of each mass, relative to the center of mass, is the same kind of conic for each.

***EXERCISE 14.2.** Starting with Eq. (14.2) for the relative motion of two particles, study the behavior of r at an instant of collision. Notice that (7.1) applies with $c = 0$, $\mu = G(m_1 + m_2)$, so that $r\dot{r}^2 = 2(\mu + hr)$. Since $r \to 0$ at a collision we have $r\dot{r}^2 \to 2\mu$ when $t \to t_1$, the time of collision. This is independent of the sign of h. Conclude that $|\dot{r}|r^{1/2} \to \sqrt{2\mu}$ and hence that $r|t - t_1|^{-2/3} \to (9/2\mu)^{1/3}$ as $t \to t_1$.

FIGURE 8. from section 1.14