HW 4, CELESTIAL MECHANICS, W 2017, UCSC. DUE MONDAY, MARCH 6 (AT THE EARLIEST)

The Newtonian potential for the standard 3-body problem is $V(q) = -G(\frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_1 m_3}{r_{13}})$. Here I ask you to change the potential to

$$V_\alpha = -G(\alpha)(\frac{m_1 m_2}{(r_{12})^\alpha} + \frac{m_2 m_3}{(r_{23})^\alpha} + \frac{m_1 m_3}{(r_{13})^\alpha})$$

where $\alpha \in \mathbb{R}$ is any exponent except 0, and investigate how some of the results derived this monday in class change.

Notation: $r_{ab} = \|q_a - q_b\|$ denotes the distance between body a and body b, both modelled as point masses. The $m_a > 0$ are the masses. The new "gravitational constant" $G(\alpha)$ is mostly just there to make certain that the units of V_α are those of energy.

A) [2 pts] What are the new 'Newton's equations"? That is, what are the inter-body forces $\vec{F}_{ab} = \vec{F}_{ab}(q)$ occuring in the new Newton's equations $m_a \ddot{\vec{q}}_a = \vec{F}_{ab} + \vec{F}_{ac}$? (Here a, b, c is any permutation of 123.)? Here $\vec{q}_a(t) \in \mathbb{R}^3$ is the position of the body a , $a = 1, 2, 3$.

B) [2 pts] Argue that the equations can be written as $\ddot{q} = \nabla U_\alpha(q)$ where $q = (\vec{q}_1, \vec{q}_2, \vec{q}_3) \in \mathbb{E} = (\mathbb{R}^3)^3$, the gradient ∇ is with respect to the mass metric $\langle \cdot, \cdot \rangle$ as described in class,and where $U_\alpha = -V_\alpha$.

C) [2 pts] Show that the energy $E = K + V_\alpha$ is conserved, making sure to describe K.

D) [10 pts] Let $I(q) = \langle q, q \rangle = m_1 \|\vec{q}_1\|^2 + m_2 \|\vec{q}_2\|^2 + m_3 \|\vec{q}_3\|^2$ be the moment of inertia as before. Derive the new Lagrange-Jacobi identity $\frac{d^2 I}{dt^2} = aE + bU_\alpha$ where a, b are constants possibly depending on the exponent α.

E) [4 pts] Something special happens to these Lagrange-Jacobi equation when $\alpha = 2$. What is it? Use this special property to solve for the time evolution of $I(t)$ in this case.