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NEW METHODS FOR THE PROBLEM OF COLLECTIVE RUIN*

CRAIG STEVEN PETERS† AND MARC MANGEL‡

Abstract. The problem of “collective ruin” arises in a number of different situations in operations research and is particularly well suited as a model of risk business such as an insurance company. The problem of collective ruin is formulated in terms of dynamical stochastic processes for a risk reserve $Z(t)$. The reserve grows according to a deterministic process $\beta(Z(t))$, the insurance premiums, and is decremented according to a compound stochastic process, claims. The integral-differential-difference equation is derived for the probability of survival to time t and a number of different methods for the solution of the stationary version of the equation, which corresponds to probability of surviving forever, are described. In particular, asymptotic techniques are developed based on the WKB method and its extensions for the solution of a broad class of risk problems. This greatly extends the classical work of Feller, Cramer, and others who were only able to treat the case in which $\beta(Z(t))$ is constant.

Key words. asymptotic approximations, WKB method, turning point, collective ruin

AMS(MOS) subject classifications. 34E20, 44A10, 45J05, 60J75, 90B99

1. Introduction. The collective ruin problem concerns the state of a risk business such as an insurance company. Seal [12] characterises the essential properties of a risk business as a risk reserve, a premium income, and an outgo of claims. For example, consider the following simple model for the operation of an insurance company. An initial sum of capital is set aside. It is augmented by the collection of premiums from policy holders and depleted by the payment of claims against the policies insured. The sum of this initial deposit plus the total premiums collected minus the total of all claims paid is called the risk reserve of an insurance company. Let $Z^*(t)$ denote the amount of the risk reserve at time t . The problem of collective ruin is then the computation of the probability that $Z^*(t) \geq 0$ for either a finite time $0 \leq t \leq s$ or for the infinite interval $t \geq 0$. Although the problem has been worked on by luminaries such as Cramer [3] and Feller [4], many questions remain concerning the solution of the problem—as well as formulations of the various extensions. Citations to more recent work can be found in papers of Asmussen [1], [2] and Siegmund [13]–[15].

Assume for simplicity that the premiums collected over a time interval of dt equal βdt units of money. It is possible that β could depend on time and the current value of the risk reserve. A fledgling company might increase its premiums over time starting with low premiums to attract new customers and then gradually increase its rates to market values. The premiums and hence β could depend upon the amount in the risk reserve. If $Z^*(t)$ falls slowly but steadily with time it might be an indication that the risk reserve will need more income, i.e., a larger β , to survive over the long run.

The occurrence of a claim corresponds to the unforeseen (though not entirely unexpected) loss to a policy holder. The uncertainty of a claim includes a random occurrence time and a random size. To separate the random elements of the claim process let $C_t = \{t_i | \text{a claim occurs at } t_i < t\}$ and X_t equal the size of a claim occurring at t . We assume that the number of claim occurrence time in C_t , referred to as the

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claims number process and denoted N_t , is a Poisson process with parameter λ . The claim sizes X_i will have a distribution function $F(x)$ (i.e., $P(X_i \leq x) = F(x)$).

If S_t denotes the accumulated claims process, then

$$(1.1) \quad S_t = \sum_{i=1}^{N_t} X_i$$

is a compound Poisson process, (see Karlin and Taylor [6]). The risk reserve now can be written

$$(1.2) \quad Z^*(t) = z + \int_0^t \beta(Z^*(\tau)) d\tau - S_t$$

where $Z^*(0) = z$. The dynamics are represented by the stochastic differential equation

$$(1.3) \quad Z^*(t+dt) - Z^*(t) = dZ^*(t) = \beta(Z^*(t)) dt - y d\pi$$

where

$$(1.4) \quad d\pi = \begin{cases} 1 & \text{with probability } \lambda dt + o(dt) \\ 0 & \text{with probability } 1 - \lambda dt + o(dt) \end{cases}$$

and y is a random variable with distribution function $F(x)$. The interpretation of (1.3) is that an incremental change in $Z^*(t)$ involves a certain income of βdt units of money and with probability λdt a payout of a claim of y units of money where y is unknown but is drawn from a distribution function F .

In this paper, our main interest will be the computation of the probability that the risk reserve remains nonnegative. In the next section we discuss the deterministic flow of the risk process, since this will guide our thinking about formulation and solution of the full problems. In § 3, we formulate the problem for the probability of survival. After scaling we are lead to partial differential integral equations for the quantities of interest. We solve these equations in §§ 4 and 5 using WKB or ray methods (Keller [5]) and generalized ray methods (Mangel and Ludwig [7], Mangel [8]), respectively. In § 6, we conclude with a discussion and some directions for future research.

2. Deterministic risk flows. In this section, we elucidate the average of the risk reserve process. To do this, we average the dynamics of (1.3) with respect to the Poisson process (claim occurrence times in the risk reserve analogy) and with respect to the size of the independent random variables (the size of the claims in the risk reserve analogy). We obtain the following dynamics for the average and thus deterministic process, Z_d ,

$$(2.1) \quad dZ_d(t) = E_y\{\beta(Z^*(t))\} dt - \lambda\mu dt$$

where

$$Z_d(t) = E_{d\pi} E_y\{Z^*(t)\}$$

$$E_{d\pi}\{y d\pi\} = y\lambda dt$$

$$E_y\{y\lambda dt\} = \mu\lambda dt.$$

Dividing (2.1) by dt and letting dt tend towards zero gives the following differential equation describing the deterministic process:

$$(2.2) \quad \frac{dZ_d}{dt} = E_{d\pi} E_y\{\beta(Z^*(t))\} - \lambda\mu.$$

If $\beta(Z^*)$ is nonlinear then in general $E_{d\pi}E_y\{\beta(Z^*)\} \neq \beta(E_{d\pi}E_y\{Z^*\})$, but if $\beta(Z^*)$ is linear, as it will be in the two principal cases investigated here, (2.2) may be written

$$(2.3) \quad \frac{dZ_d}{dt} = \beta(Z_d) - \lambda\mu.$$

For example, if $\beta(Z^*)$ is identically a constant β then (2.3) is simply

$$(2.4) \quad \frac{dZ_d}{dt} = \beta - \lambda\mu.$$

Interpretations of β , λ , μ will be given later, but here they are assumed to be positive. If $\beta < \lambda\mu$ then $dZ_d/dt < 0$ and the deterministic process is drawn inexorably toward zero, and therefore the probability of ultimate survival for the actual process $Z^*(t)$ is intuitively zero. If $\beta > \lambda\mu$ then $dZ_d/dt > 0$, then the deterministic process is driven away from the origin. Thus intuitively the possibility exists for $Z^*(t)$ to remain nonnegative forever. This is by no means certain since the actual process $Z^*(t)$ ends if it ever falls below zero.

The situation for general $\beta(z)$ can be illustrated by considering the specific situation $\beta(Z^*) = \beta + \gamma Z^*$. In this case, (2.3) becomes

$$(2.5) \quad \frac{dZ_d}{dt} = (\beta - \lambda\mu) + \gamma Z_d.$$

If $\beta > \lambda\mu$ then (for $\gamma > 0$), then the deterministic system does not have a rest point.

If $\beta < \lambda\mu$, then differential equation (2.5) has a positive rest point, $Z_{d,rest}$, given by

$$(2.6) \quad Z_{d,rest} = \frac{1}{\gamma}(\lambda\mu - \beta).$$

If $\beta > \lambda\mu$ then the phase line of (2.5) is shown in Fig. 1(a). The phase line of the deterministic process is referred to as the deterministic risk flow. Deterministic risk flow as depicted in Fig. 1(a) shall be referred to as case A. For $\beta < \lambda\mu$, the phase line of (2.5) is shown in Fig. 1(b). Deterministic risk flow as depicted in Fig. 1(b) shall be referred to as case B. To the right of this rest point we expect the real process to behave much as it did in case A because the flow of the differential equation (2.6) is away from the rest point, though a large enough claim could bring $Z^*(t)$ below $Z_{d,rest}$. To the left of the rest point, matters are different, the principal difference being that $0 < Z^*(t) < Z_{d,rest}$ is not an absorbing state for case B flows. To the left of $Z_{d,rest}$, the flow of the average process is toward the origin but now $Z^*(t)$ can move against the flow and even exceed $Z_{d,rest}$ by a series of nonevents (i.e., claims not occurring).

3. Survival and extinction of the risk process. To characterize survival of the risk process, we define $R(z, t)$ as the probability that the risk reserve remains nonnegative through time t . That is, let

$$(3.1) \quad R(z, t) = \Pr \{Z^*(s) \geq 0 \text{ for all } 0 \leq s \leq t \mid Z^*(0) = z\}.$$

We can think of $R(z, t)$ as the probability that the sum of independent random variables, S_t , does not exceed the barrier Z_b where Z_b satisfies

$$(3.2) \quad \frac{dZ_b}{dt} = \beta(Z(t)).$$

If $\beta(Z(t)) = \beta$, a constant, then the solution to (3.2) is simply $Z_b = \beta t + z$ where $Z_b(0) = z$. The barrier need not grow as a linear function of $Z(t)$. For example, if

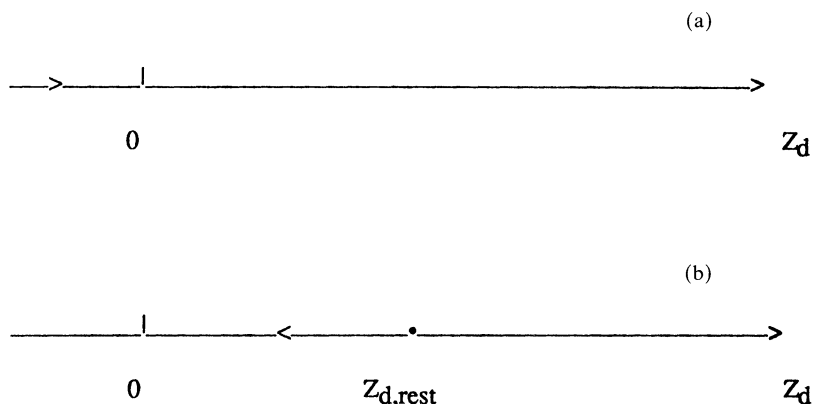


FIG. 1. The deterministic risk flows. We show the phase lines of interest for the averaged system $dZ_d/dt = \beta(Z_d) - \lambda\mu$. In case A (panel (a)), the deterministic flow is always towards the right, so that the stochastic fluctuations and deterministic flow are in opposing directions. In case B (panel (b)), the deterministic flow has an unstable rest point to the right of the origin. We denote this point by $Z_{d,rest}$. The deterministic flow is towards increasing Z_d if $z > Z_{d,rest}$ and towards the origin otherwise.

$\beta(Z(t)) = \beta + \gamma Z(t)$, where β and γ are constants, (3.2) has the solution $Z_b(t) = -\beta/\gamma + (z + \beta/\gamma)e^{\gamma t}$.

Before deriving an equation for $R(z, t)$, we scale $Z^*(t)$ by writing $Z^*(t) = kZ(t)$. The rationale is that in practice one might expect an insurance company to deal with large sums of money in each transaction. A typical value of k might be 10,000. Let $\varepsilon = 1/k$; then the risk reserve dynamics become

$$(3.3) \quad dZ = \varepsilon(\beta(Z(t)) dt - \gamma d\pi).$$

As ε tends toward zero so does dZ , and thus in the limit neither the claims nor the premiums have an appreciable effect on the risk reserve. This makes sense since as ε tends toward zero, k tends toward infinity and the size of the original risk reserve $Z^*(t)$ becomes infinite. Hence we will construct a large Z solution.

$R(z, t)$ satisfies the following master equation, obtained by applying the law of total probability

$$(3.4) \quad R(z, t) = E_{dZ}\{R(z + dZ, t - dt)\}$$

where E_{dZ} is the expected value operator over all possible jumps dZ of Z . For convenience we suppress the dependence of β on $Z(t)$ and note that the stochastic differential dZ has only two values, either $\varepsilon\beta dt - \varepsilon\gamma$ or $\varepsilon\beta dt$, making the computation of E_{dZ} particularly simple. Thus (3.4) can be written

$$(3.5) \quad \begin{aligned} R(z, t) = & (1 - \lambda dt)R(z + \varepsilon\beta dt, t - dt) \\ & + \lambda dt \int_0^{z/\varepsilon + \beta dt} R(z + \varepsilon\beta dt - \varepsilon y, t - dt) dF(y) + o(dt). \end{aligned}$$

Taylor expanding, for small dt with ε fixed, $R(z + \varepsilon\beta dt, t - dt)$ to order dt and simplifying yields

$$(3.6) \quad R_t = -\lambda R + \varepsilon\beta dt R_z + \lambda dt \int_0^{z/\varepsilon + \beta dt} R(z + \varepsilon\beta dt - \varepsilon y, t - dt) dF(y) + o(dt).$$

Dividing by dt and letting $dt \rightarrow 0$ gives

$$(3.7) \quad R_t(z, t) = -\lambda R(z, t) + \varepsilon \beta R_z(z, t) + \lambda \int_0^{z/\varepsilon} R(z - \varepsilon y, t) dF(y).$$

The presence of a derivative indicates that a solution of (3.7) will contain one arbitrary constant. To determine this constant add the condition that

$$(3.8) \quad \lim_{z \rightarrow \infty} R(z, t) = 1.$$

Equation (3.8) indicates that for an infinite risk reserve the probability of survival is one. In addition, we have the interval condition that $R(z, t) = 0$ if $z < 0$.

We define the probability of ultimate survival $R(z)$ by

$$\lim_{t \rightarrow \infty} R(z, t) = R(z).$$

The integro-differential equation for $R(z)$ is

$$(3.9) \quad 0 = -\lambda R(z) + \varepsilon \beta(z) R'(z) + \lambda \int_0^{z/\varepsilon} R(z - \varepsilon y) dF(y).$$

By analogy to (3.8), we have

$$(3.10) \quad \lim_{z \rightarrow \infty} R(z) = 1$$

and the interval condition $R(z) = 0$ if $z < 0$. The value of $R(0)$ must be determined as part of the solution of (3.9).

In the next section we show that if $\beta(Z)$ is constant, (3.9) can be solved by means of the Laplace transform. However, for general $\beta(Z)$, the method of solution by Laplace transform does not work. The literature on risk theory (e.g., Seal [12]) essentially treats only the constant coefficient problem, even though this is highly unrealistic. Our contribution will be to provide a method that can be used to solve (3.9) for arbitrary $\beta(Z)$.

We now turn to the asymptotic solution of (3.9). The form of the solution depends upon the flow of the deterministic system.

4. Asymptotic solution when the flow is always to the right (case A). We motivate the form of the asymptotic solution by the study of special cases: i) solution by Laplace transform and ii) a case in which we can convert the integral equation to an ordinary differential equation.

4.1. Solution of the constant coefficient problem by Laplace transforms. One way of solving (3.9) for constant β is by Laplace transforms. To do this, set $\eta = z/\varepsilon$; (3.9) becomes

$$(4.1) \quad 0 = -\lambda R(\varepsilon \eta) + \beta R'(\varepsilon \eta) + \lambda \int_0^\eta R(\varepsilon \eta - \varepsilon y) dF(y).$$

Set $r(\eta) = R(\varepsilon \eta)$; then (4.1) becomes

$$(4.2) \quad 0 = -\lambda r(\eta) + \beta r'(\eta) + \lambda \int_0^\eta r(\eta - y) dF(y).$$

For an absolutely continuous claim distribution function $F(x)$ the Laplace transform of this equation is

$$(4.3) \quad 0 = -\lambda r^\wedge(s) + \beta[r^\wedge(s) - r(0)] + \lambda r^\wedge(s) F^\wedge(s)$$

where

$$(4.4) \quad \begin{aligned} r^\wedge(s) &= \int_0^\infty e^{-s\eta} r(\eta) d\eta, & F^\wedge(s) &= \int_0^\infty e^{-sy} \frac{dF(y)}{dy} dy \\ \int_0^\infty e^{-s\eta} r'(\eta) d\eta &= sr^\wedge(s) - r(0). \end{aligned}$$

Solving for $r^\wedge(s)$ yields

$$(4.5) \quad r^\wedge(s) = \frac{-\beta r(0)}{\lambda - \lambda F^\wedge(s) - \beta s}.$$

In order to invert $r^\wedge(s)$ to recover $r(\eta)$ we need to know both $r(0)$ and $F^\wedge(s)$. Consider $r(0)$ first. The final-value theorem in the theory of Laplace transforms [16] shows that

$$\lim_{s \rightarrow 0} sr^\wedge(s) = \lim_{\eta \rightarrow \infty} r(\eta).$$

For this problem $\lim_{\eta \rightarrow \infty} r(\eta) = 1$, so that

$$(4.6) \quad \lim_{s \rightarrow 0} sr^\wedge(s) = \lim_{s \rightarrow 0} \frac{-s\beta r(0)}{\lambda - \lambda F^\wedge(s) - \beta s} = 1.$$

We use l'Hopital's rule once to evaluate the limit in (4.6) and obtain

$$(4.7) \quad r(0) = 1 - \frac{\lambda}{\beta} \mu$$

where μ is the first moment of $F(x)$.

The parameter λ has the interpretation as the average number of claims per unit time. Since μ is the average cost of a claim, $\lambda\mu$ is the average payout per unit time; β is the average income per unit time so that $\lambda\mu/\beta$ is the ratio of an insurance company's average payout per unit time to its average income per unit time. If $\lambda\mu/\beta$ is greater than 1, then we intuitively expect that the insurance company will eventually be ruined.

With $r(0)$ known and $F^\wedge(s)$ expressible in the form of an integral, (4.5) can be inverted in the general case of an arbitrary claims distribution by expressing $r(\eta)$ in terms of (Spiegel [16]) Bromwich's integral formula as

$$(4.8) \quad r(\eta) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{s\eta} r^\wedge(s) ds \quad \eta > 0$$

where σ is a real number chosen so that the complex integration is performed along the line $s = \sigma$ lying to the right of all singularities such as poles, branch points, or essential singularities.

For some distribution functions, $F^\wedge(s)$ can be found explicitly by performing the integration analytically. A robust class of distributions for which this is possible is the gamma distribution, i.e.,

$$dF(x) = \frac{\alpha^\nu}{\Gamma(\nu)} e^{-\alpha x} x^{\nu-1} \quad \text{and} \quad \Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t} dt.$$

Specific solutions will be given for $\nu = 1$ and $\nu = 2$.

For $\nu = 1$ the gamma distribution reduces to an exponential distribution and $dF(x) = \alpha e^{-\alpha x} dx$ and $F(x) = 1 - e^{-\alpha x}$. For the exponential distribution

$$(4.9) \quad \begin{aligned} F^\wedge(s) &= \frac{\alpha}{s + \alpha} \\ \mu &= \frac{1}{\alpha} \end{aligned}$$

so that

$$(4.10) \quad r(0) = 1 - \frac{\lambda}{\beta\alpha}.$$

Then $r^\wedge(s)$ given by (4.5) becomes

$$(4.11) \quad r^\wedge(s) = \frac{1}{s} - \frac{\lambda}{\beta\alpha} \frac{1}{s + (\alpha - \lambda/\beta)}.$$

The inverse Laplace transform of $r^\wedge(s)$ is

$$(4.12) \quad r(\eta) = 1 - \frac{\lambda}{\beta\alpha} e^{-(\alpha - \lambda/\beta)\eta}.$$

That (4.12) is the solution of (4.2) can be verified by direct substitution into (4.2).

Replacing η by z/ε , it is obvious from the form of (4.12) that as $\varepsilon \rightarrow 0$ there exists a boundary layer near the origin. For $\nu = 2$, we find that

$$(4.13) \quad \begin{aligned} F^\wedge(s) &= \frac{\alpha}{(\alpha + s)^2} \\ \mu &= \frac{2}{\alpha} \end{aligned}$$

so that

$$(4.14) \quad r^\wedge(s) = \frac{-\beta(1 - 2\lambda/\beta\alpha)(\alpha^2 + 2\alpha s + s^2)}{s[(2\lambda\alpha - \beta\alpha^2) + (\lambda - 2\beta\alpha)s - \beta s^2]}.$$

After a partial fraction decomposition and inverse Laplace transform, we obtain

$$(4.15) \quad \begin{aligned} r(\eta) &= 1 - \frac{\lambda}{2\beta} e^{(\lambda/2\beta - \alpha)\eta} \left[\cosh(\sqrt{(\lambda^2/4\beta^2) + (\lambda\alpha/\beta)} \eta) \right. \\ &\quad \left. + \frac{(\lambda/2\beta) + (\alpha/2)}{\sqrt{(\lambda^2/4\beta^2) + (\lambda\alpha/\beta)}} \sinh(\sqrt{(\lambda^2/4\beta^2) + (\lambda\alpha/\beta)} \eta) \right]. \end{aligned}$$

For large η , we obtain

$$(4.16) \quad r(\eta) \sim 1 - \frac{\lambda}{\beta\alpha} \exp\left(\left(\frac{\lambda}{2\beta} - \alpha\right) + \sqrt{\frac{\lambda^2}{4\beta^2} + \frac{\lambda\alpha}{\beta}} \eta\right) \left[1 + \frac{(\lambda/2\beta) + (\alpha/2)}{(\lambda^2/4\beta^2) + (\lambda\alpha/\beta)}\right].$$

Note that both (4.12) and (4.16) take the form

$$(4.17) \quad r(\eta) \sim 1 - k(\eta) e^{-\psi(\eta)}$$

for appropriate choices of $k(\eta)$ and $\psi(\eta)$.

In choosing parameter values we are restricted by the condition that $r(0) = 1 - 2\lambda/\beta\alpha > 0$; see § 2. Therefore if a company models the claim size distribution with

a gamma distribution $\nu = 2$ it will necessarily require more income (i.e., larger β) than if it modeled its claim size distribution with an exponential distribution where the requirement is $r(0) = 1 - \lambda/\beta\alpha > 0$. This is to be expected since the mean of the claim sizes doubles from $1/\alpha$ to $2/\alpha$.

4.2. Conversion to an ordinary differential equation. If ν in the gamma density is an integer, another way of solving (3.9) is to convert it from an integral equation to a differential equation (Knessl et al. [7]). When we do this, a differential equation of order $\nu + 1$ is obtained. The method will be illustrated for the exponential claim distribution.

To employ this method, we must return to (4.1), which we multiply by $e^{\alpha\eta}$ to obtain

$$(4.18) \quad 0 = -\lambda r(\eta) e^{\alpha\eta} + \beta r'(\eta) e^{\alpha\eta} + \lambda \int_0^\eta r(\eta - y) e^{\alpha(\eta - y)} dy.$$

Now differentiate with respect to η

$$(4.19) \quad 0 = -\lambda r(\eta)\alpha e^{\alpha\eta} - \lambda r'(\eta) e^{\alpha\eta} + \beta\alpha r'(\eta) e^{\alpha\eta} + \beta r''(\eta) e^{\alpha\eta} + \frac{d}{d\eta} \left[\lambda \int_0^\eta r(\omega) e^{\alpha\omega} d\omega \right].$$

Collecting terms gives

$$(4.20) \quad 0 = r''(\eta) + \left(\alpha - \frac{\lambda}{\beta} \right) r'(\eta).$$

There are two undetermined coefficients, so two conditions are needed to uniquely specify the solution. One solution is $r(\eta) \rightarrow 1$ as $\eta \rightarrow \infty$. The other is obtained by setting $\eta = 0$ in equation (4.5), yielding $r'(0) = (\lambda/\beta)r(0)$. Hence $r(\eta)$ can be written

$$(4.21) \quad r(\eta) = 1 - \frac{\lambda}{\beta\alpha} \exp \left(- \left(\alpha - \frac{\lambda}{\beta} \right) \eta \right).$$

Note that this method can be used if β is nonconstant (but the distribution is a gamma with integer parameter); the differential equation (4.20) will change but the method is still appropriate.

For example, if we consider the case in which $\beta(z) = \beta + \gamma z$, then (4.19) is replaced by

$$(4.22) \quad 0 = -\lambda r'(\eta) e^{\alpha\eta} - \lambda r(\eta)\alpha e^{\alpha\eta} + (\beta + \gamma\eta)r''(\eta) e^{\alpha\eta} + \gamma\eta r'(\eta) e^{\alpha\eta} + \alpha(\beta + \gamma\eta)r'(\eta) e^{\alpha\eta} + \lambda\alpha \frac{d}{d\eta} \left[\int_0^\eta r(\omega) e^{\alpha\omega} d\omega \right].$$

Differentiating and collecting terms yields

$$(4.23) \quad r''(\eta) + \left[\alpha + \frac{\gamma\eta - \lambda}{\beta + \gamma\eta} \right] r'(\eta) = 0$$

which has the general solution

$$(4.24) \quad r(\eta) = r'(0)\beta^{1-(\lambda/\gamma\epsilon)} \int_0^\eta \text{jtePe}^{-\alpha s} (\beta + \gamma\epsilon s)^{(\lambda/\gamma\epsilon)-1} ds + r(0).$$

Using the same two conditions as before leads to the following system for $r'(0)$ and $r(0)$

$$\begin{aligned} 1 &= r'(0)\beta^{1-(\lambda/\gamma\epsilon)} \int_0^\infty e^{-\alpha s} (\beta + \gamma\epsilon s)^{(\lambda/\gamma\epsilon)-1} ds + r(0) \\ 0 &= -\lambda r(0) + \beta r'(0). \end{aligned}$$

Solving for $r'(0)$ and $r(0)$ yields the following solution for $r(\eta)$

$$(4.25) \quad r(\eta) = \frac{\lambda \beta^{-(\lambda/\gamma\epsilon)} \int_0^\eta e^{-\alpha s} (\beta + \gamma \epsilon s)^{\lambda/\gamma\epsilon - 1} ds + 1}{\lambda \beta^{-(\lambda/\gamma\epsilon)} \int_0^\infty e^{-\alpha s} (\beta + \gamma \epsilon s)^{\lambda/\gamma\epsilon - 1} ds + 1}.$$

4.3. WKB solution for the probability of survival. We will now show how the general problem (3.9)—for arbitrary $\beta(z)$, but deterministic risk flow from the origin (case A)—can be solved by WKB method if ϵ is small. To begin, since $R(z) = 0$ for $z < 0$, we can extend the region of integration in (3.9) to infinity. This gives

$$(4.26) \quad 0 = -\lambda R(z) + \epsilon \beta(z) R'(z) + \lambda \int_0^\infty R(z - \epsilon y) dF(y).$$

Based on (4.12) and (4.16), we seek a solution of (4.26) in the form

$$(4.27) \quad R(z) \sim 1 - k(z) e^{\psi(z)/\epsilon}$$

where $k(z) = \sum_{i=0}^\infty k_i(z) \epsilon^i$ and $\psi(z)$ must be determined. Here we will explicitly derive and solve the equations satisfied by $\psi(z)$ and $k_0(z)$. The functions $k_i(z)$ for $i \geq 1$ are obtained in a similar fashion. Substituting (4.27) into (4.26) gives

$$(4.28) \quad \begin{aligned} 0 = & -\lambda (1 - k_0 e^{\psi/\epsilon}) + \epsilon \beta(z) \left(-k'_0 - \frac{k_0 \psi'}{\epsilon} \right) e^{\psi/\epsilon} \\ & + \lambda \int_0^\infty [1 - k_0(z - \epsilon y) e^{\psi(z - \epsilon y)/\epsilon}] dF(y). \end{aligned}$$

Now expand $k_0(z - \epsilon y)$ in a Taylor series for small ϵ to obtain

$$(4.29) \quad \begin{aligned} 0 = & -\lambda (1 - k_0 e^{\psi/\epsilon}) + \epsilon \beta(z) \left(-k'_0 - \frac{k_0 \psi'}{\epsilon} \right) e^{\psi/\epsilon} \\ & + \lambda \int_0^\infty [1 - (k_0 - \epsilon y k'_0) e^{\psi/\epsilon - y\psi' + y^2(\epsilon/2)\psi''}] dF(y) + O(\epsilon). \end{aligned}$$

Setting the coefficient of $e^{\psi(z)/\epsilon}$ equal to zero gives the eikonal equation (Keller [5])

$$(4.30) \quad 0 = \lambda - \beta(z) \psi'(z) - \lambda \int_0^\infty e^{-y\psi'(z)} dF(y).$$

The moment generating function for the random variable Y is given by

$$(4.31) \quad M(\psi'(z)) \equiv \int_0^\infty e^{-y\psi'(z)} dF(y) = E\{e^{-Y\psi'(z)}\}$$

so that the eikonal equation can be written as

$$(4.32) \quad \frac{\beta(z)}{\lambda} \psi'(z) = 1 - M(\psi'(z)).$$

Thus, we solve a nonlinear first-order equation for $\psi(z)$. Clearly $\psi'(z) = 0$ is a solution of this equation, which we reject. For the cases we consider, there is one positive solution of (4.32); there may be negative solutions, which we also reject.

Setting the coefficient of $\varepsilon e^{\psi/\varepsilon}$ in (4.29) equal to 0 gives the first transport equation:

$$(4.33) \quad 0 = k'_0(z) \left[\lambda \int_0^\infty y e^{-y\psi'(z)} dF(y) - (\beta + \gamma z) \right] \\ - k_0(z) \left[\frac{\lambda \psi''(z)}{2} \int_0^\infty y^2 e^{-y\psi'(z)} dF(y) \right]$$

which has solution

$$(4.34) \quad k_0(z) = \kappa \exp \left(\int_0^z p(\xi) d\xi \right)$$

where κ is a constant and

$$(4.35) \quad p(\xi) = \frac{(\lambda \psi''(\xi))/2 \int_0^\infty y^2 e^{-y\psi'(\xi)} dF(y)}{\lambda \int_0^\infty y e^{-y\psi'(\xi)} dF(y) - (\beta + \gamma \xi)}.$$

To leading order, we have

$$(4.36) \quad R(z) \sim 1 - k_0(z) e^{(\psi(z))/\varepsilon}.$$

This solution involves two unknowns— κ and the integration constant from (4.34). To find these, we match using the method of matched asymptotic expansions (Nayfeh [10]) to the inner solution obtained by solving (4.26) with z frozen around 0. We will illustrate the matching procedure with two examples, both using $\beta(z) = \beta + \gamma z$. First consider the exponential claims distribution. We find that the eikonal function is

$$(4.37) \quad \psi(z) = \frac{\lambda}{\gamma} \ln(\beta + \gamma z) - \alpha z + c$$

where c is a constant of integration. The solution to (4.33) is

$$(4.38) \quad k_0(z) = \kappa \left[\frac{\beta - \lambda/\alpha}{\beta + \gamma z - \lambda/\alpha} \right].$$

Thus to leading order, the outer solution is then

$$(4.39) \quad R(z) = 1 - k \left[\frac{\beta - \lambda/\alpha}{\beta + \gamma z - \lambda/\alpha} \right] \exp \left(\frac{\lambda}{\gamma \varepsilon} \ln(\beta + \gamma z) - \alpha \frac{z}{\varepsilon} + \frac{c}{\varepsilon} \right).$$

The inner solution of (4.26) is (from § 4.2)

$$(4.40) \quad r(\eta) = 1 - \frac{\lambda}{\beta \alpha} \exp \left(\left(\frac{\lambda}{\beta} - \alpha \right) \eta \right).$$

The standard matching procedure shows that we should choose

$$\kappa = \frac{\lambda}{\beta \alpha} \quad \text{and} \quad c = -\frac{\lambda}{\gamma \varepsilon} \ln(\beta).$$

Hence the solution of (4.26) for this case is

$$(4.41) \quad R(z) = 1 - \frac{\lambda}{\beta \alpha} \left[\frac{\beta - \lambda/\alpha}{\beta + \gamma z - \lambda/\alpha} \right] \exp \left(\frac{\lambda}{\gamma \varepsilon} \ln \left(1 + \frac{\gamma z}{\beta} \right) - \alpha \frac{z}{\varepsilon} \right).$$

Numerical comparison shows that there is not much difference between (4.41) and (4.12) for $\gamma = .1$.

As a second example, choose $\beta(z) = \beta + \gamma z$ still, but let $F(y)$ be a gamma density with $\nu = 2$. Then the eikonal equation becomes

$$(4.42) \quad \psi'(z) = \frac{\lambda}{\beta + \gamma z} \left[1 - \frac{\alpha^2}{(\alpha + \psi'(z))^2} \right]$$

which has solution

$$(4.43) \quad \psi(z) = \frac{1}{2} \int_0^z \left[\frac{\lambda}{\beta + \gamma \omega} - 2\alpha \pm \sqrt{\left[\frac{\lambda}{\beta + \gamma \omega} \right]^2 + \frac{4\alpha\lambda}{\beta + \gamma \omega}} \right] d\omega + \psi(0).$$

In (4.43) the negative square root is rejected since it leads to a divergent integral in (4.30). The outer solution is

$$(4.44) \quad R(z) = 1 - K \exp \left(\frac{1}{2\varepsilon} \int_0^z \left[\frac{\lambda}{\beta + \gamma \omega} - 2\alpha + \sqrt{\left[\frac{\lambda}{\beta + \gamma \omega} \right]^2 + \frac{4\alpha\lambda}{\beta + \gamma \omega}} \right] d\omega + \int_0^z p(\xi) d\xi \right)$$

where $K = \kappa e^{\psi(0)/\varepsilon}$ and $p(\xi)$ is defined as before. When we perform the matching, we find that

$$(4.45) \quad K = \frac{\lambda}{\beta\alpha} \left[1 + \frac{\lambda/2\beta + \alpha/2}{\sqrt{\lambda^2/4\beta^2 + \lambda\alpha/\beta}} \right].$$

For the parameter values $\lambda = 1$, $\beta = 2.01$, $\alpha = 1$, $\gamma = .1$, $\varepsilon = .1$, we find that the difference in survival probability for the models with and without interest can be highly significant for the gamma distribution with $\nu = 2$. For example, at $z = 1$ the former is .99 versus .6 in the latter, and at $z = .25$, the former is .7 versus .2 in the latter. This shows the importance of developing methods that can be used to deal with the case of nonconstant coefficients. We believe that this is especially true if we want to deal with claim distributions more complicated than the exponential. Another way to think of the difference between $\nu = 1$ and $\nu = 2$ is that for the exponential distribution the WKB expansion reduces to the boundary layer expansion for small z . Thus, there really is no boundary layer when $\nu = 1$. On the other hand, there is a significant boundary layer when $\nu = 2$.

For an arbitrary claims distribution and arbitrary $\beta(z)$ an explicit solution such as (4.44) is not always possible, yet a numerical evaluation of the asymptotic solution is quite feasible. In this general case, (4.32) is solved numerically for $\psi(z)$ and this in turn is used in the numerical solution of (4.33). The matching is performed in the same fashion but only a numerical value of K is obtained. In the past, authors such as Seal [12], Cramer [3], and Feller [5] have solved common cases such as an exponential claims distribution and income proportional to time (i.e., $\beta(z)$ identically a constant). The WKB method has provided a procedure for determining an asymptotic solution to (3.9) for a much richer class of models for risk reserve dynamics.

5. Asymptotic solution for an interior rest point (case B). We now construct the asymptotic solution of (3.9) when the deterministic dynamics have an interior rest point. For this case, (4.33) has a singularity when the coefficient of $k'_0(z)$ vanishes. We call the value of z at which this coefficient vanishes a turning point and is denoted z_{turn} . The WKB solution breaks down (Nayfeh [10]) near the turning point. To motivate our asymptotic solution, we once again study the case of the exponential claims

distribution with $\beta(z) = \beta + \gamma z$. From (4.38) we see that the solution to the transport equation fails to exist near

$$(5.1) \quad z_{\text{turn}} = \frac{1}{\gamma} \left(\frac{\lambda}{\alpha} - \beta \right).$$

By comparing (5.1) and (4.10) we see that if $z_{\text{turn}} > 0$ then $r(0) < 0$. If $r(0) < 0$ then $\lambda/\beta\alpha > 1$ and $e^{-(\alpha-\lambda/\beta)\eta} \geq 1$ for all $\eta \geq 0$ and thus $r(\eta)$ as given in (4.12) is strictly less than zero for all $\eta \geq 0$. We conclude that the solution given by (4.12) is meaningless. For the exponential claims distribution, however, we can use the method of § 4.2 to convert the integral equation to an ordinary differential equation. When this is done for arbitrary $\beta(z)$ we obtain

$$(5.2) \quad \varepsilon R''(z) + h(z) R'(z) = 0$$

where

$$(5.3) \quad h(z) = \left[\alpha - \frac{\lambda}{\beta(z)} + \frac{\varepsilon \beta'(z)}{\beta(z)} \right].$$

The differential equation (5.2) has the boundary condition of $R(z) \rightarrow 1$ as $z \rightarrow \infty$. To construct the leading term in the asymptotic expansion of the solution, and to motivate our subsequent work, we add the condition that $R(0) = 0$. With these boundary conditions the solution to (5.2) can be written as

$$(5.4) \quad R(z) = \frac{\int_0^z \exp \left(- \int_{z_{\text{turn}}}^{z'} \frac{h(s)}{\varepsilon} ds \right) dz'}{\int_0^\infty \exp \left(- \int_{z_{\text{turn}}}^{z'} \frac{h(s)}{\varepsilon} ds \right) dz'}.$$

From (5.4), we have that $R(0)$ is exactly 0. In general $R(0)$ is asymptotically small, i.e., there is always some chance of surviving, but $R(0) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Let z_{rest} satisfy $h(z_{\text{rest}}) = 0$. Note that $z_{\text{rest}} \neq z_{\text{turn}}$ but to $O(1)$ they are equal. Hence if the dependence of h on ε is made explicit by writing $h(z) = h(z, \varepsilon)$, then we see that $h(z_{\text{rest}}, \varepsilon) = h(z_{\text{turn}}, 0) = 0$. To determine the behavior of (5.4) near z_{turn} , we use Laplace's method to evaluate these integrals. We find that

$$(5.5) \quad R(z) \sim \frac{\int_{-\infty}^z \exp \left(- \frac{h'(z_{\text{turn}}, 0)}{2} \frac{(z' - z_{\text{turn}})^2}{\varepsilon} \right) dz'}{\int_{-\infty}^\infty \exp \left(- \frac{h'(z_{\text{turn}}, 0)}{2} \frac{(z' - z_{\text{turn}})^2}{\varepsilon} \right) dz'}.$$

Inspection of (5.5) suggests that the independent and dependent inner variables (i.e., near the turning point) are

$$(5.6) \quad \eta = \frac{(z - z_{\text{turn}})}{\sqrt{\varepsilon}} \sqrt{h'(z_{\text{turn}}, 0)}$$

$$(5.7) \quad r(\eta) = R \left(\frac{\sqrt{\varepsilon} \eta}{\sqrt{h'(z_{\text{turn}}, 0)}} + z_{\text{turn}} \right).$$

Thus the inner solution, $r(\eta)$, may be written

$$(5.8) \quad r(\eta) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\eta} e^{-(s^2/2)} ds.$$

The scaling in (5.6) indicates that the inner region is of width $O(\varepsilon^{1/2})$.

We construct outer solutions to the right $R_r(z)$ and to the left $R_l(z)$ of the turning point. These are given by

$$(5.9) \quad R_r(z) = 1 - K_r \left[\frac{\frac{\lambda}{\alpha} - \beta}{\beta - \frac{\lambda}{\alpha} + \gamma z} \right] \exp \left(\frac{\lambda}{\gamma \varepsilon} \ln(\beta + \gamma z) - \frac{\alpha z}{\varepsilon} + \frac{c_r}{\varepsilon} \right)$$

$$(5.10) \quad R_l(z) = K_l \left[\frac{\frac{\lambda}{\alpha} - \beta}{\frac{\lambda}{\alpha} - \beta - \gamma z} \right] \exp \left(\frac{\lambda}{\gamma \varepsilon} \ln(\beta + \gamma z) - \frac{\alpha z}{\varepsilon} + \frac{c_l}{\varepsilon} \right).$$

One way to obtain a uniformly valid approximation is to match approximation (5.8) to WKB approximations (5.9) and (5.10) which determines the constants K_r , c_r and K_l , c_l , respectively. Performing the matching yields

$$c_l = c_r = -\frac{\lambda}{\gamma} \ln(\beta + \gamma z_{\text{turn}}) + \alpha z_{\text{turn}}$$

$$K_l = K_r = \frac{1}{2}.$$

This procedure, however, will not work for arbitrary claims distribution, since conversion of the integro-differential equation to an ordinary differential equation is not possible in general. The value of the canonical problem has been to suggest that the solution $R(z)$ to (4.26) is approximately given by (5.8) for z near z_{turn} . That is, based on (5.8) we assume (Mangel and Ludwig [8])

$$(5.11) \quad R(z) = \sum_{n=0}^{\infty} \varepsilon^n g_n(z) E \left(\frac{\psi(z)}{\sqrt{\varepsilon}} \right) + \varepsilon^{n+(1/2)} h_n(z) E' \left(\frac{\psi(z)}{\sqrt{\varepsilon}} \right)$$

where $\{g_n(z)\}$ and $\{h_n(z)\}$ and $\psi(z)$ are to be determined and $E(z)$ is the error function

$$(5.12) \quad E(z) = \int_{-\infty}^z e^{-(s^2/2)} ds$$

$$(5.13) \quad E''(z) + zE'(z) = 0.$$

Mangel and Ludwig [8] construct an asymptotic solution to the backward Kolmogorov equation for processes whose deterministic dynamics have a phase line identical, up to translation, to the phase line of the deterministic dynamics resulting in a turning point.

Inserting ansatz (5.11) into (4.26) the leading order terms are, after some computations,

$$(5.14) \quad -\lambda [Eg_0 + \sqrt{\varepsilon} E'h_0] + \beta [\sqrt{\varepsilon} \psi' E'g_0 + \varepsilon E'g'_0 + \sqrt{\varepsilon^3} E'h'_0 - \sqrt{\varepsilon} \psi' \psi E'h_0] \\ + \lambda \int_0^\infty \left[E - \sqrt{\varepsilon} E' \frac{1}{\psi} [1 - e^{y\psi'\psi}] + O(\sqrt{\varepsilon^3}) \right] [g_0 - \varepsilon y g'_0 + O(\varepsilon^2)] \\ + \sqrt{\varepsilon} [E' + E'[1 - e^{y\psi'\psi}] + O(\varepsilon)] [h_0 - \varepsilon y h'_0 + O(\varepsilon^2)] dF(y) = 0.$$

Collecting terms that are $O(\varepsilon^{1/2})$ and equating them to zero gives

$$(5.15) \quad (g_0 - \psi h_0) \left[\beta \psi' - \lambda \int_0^\infty \frac{1}{\psi} [1 - e^{y\psi'\psi}] dF(y) \right] = 0.$$

Thus the eikonal equation for ψ is

$$(5.16) \quad \beta(z)\psi'(z)\psi(z) - \lambda \int_0^\infty [1 - e^{y\psi'(z)\psi(z)}] dF(y) = 0.$$

Collecting terms that are $O(\varepsilon)$ and equating them to zero yields

$$(5.17) \quad [\beta(z) - \lambda\mu]g_0'(z) = 0$$

which is analogous to the transport equation. To leading order in ε the approximation (5.11) becomes

$$(5.18) \quad R(z) \sim g_0(z) \int_{-\infty}^{(\psi(z)/\sqrt{\varepsilon})} e^{-(s^2/2)} ds$$

where $\psi(z)$ is determined by (5.16) and $g_0(z)$ is determined by (5.17).

To solve (5.16) use the transformation (Mangel and Ludwig [8])

$$(5.19) \quad \phi = \frac{1}{2}\psi^2$$

to transform (5.16) into

$$(5.20) \quad \beta(z)\phi'(z) - \lambda \int_0^\infty (1 - e^{y\phi'(z)}) dF(y) = 0.$$

Once equation (5.20) is solved for ϕ , transformation (5.19) implies that

$$(5.21) \quad \psi = \pm\sqrt{2\phi}.$$

Both square roots in (5.21) are needed to describe $R(z)$ over the real line. The function ψ must pass continuously through zero as z passes through z_{turn} since $R(z)$ behaves continuously. To accomplish this, take advantage of the arbitrary constant in the solution of ϕ and choose it so that $\phi(z_{\text{turn}}) = 0$. Note that $-E(z) = E(-z)$ and that

$$(5.22) \quad g_0(z) = \begin{cases} 1/\sqrt{2\pi} & z \geq z_{\text{turn}} \\ -1/\sqrt{2\pi} & z < z_{\text{turn}} \end{cases}$$

satisfies the transport equation (5.17), since by definition $\beta(z_{\text{turn}}) - \lambda\mu = 0$. Thus the approximation (5.18), with $\psi(z)$ given by (5.21), is our leading order uniformly valid asymptotic approximation. The value of the constant for $g_0(z)$ is chosen so that $R(z) \rightarrow 1$ as $z \rightarrow \infty$. Far away from the turning point, an expansion of the error function shows that our uniform solution reduces to the solutions in (5.9) and (5.10).

For the case of an exponential distribution, we find that

$$(5.23) \quad \phi(z) = \frac{\lambda}{\gamma} \ln \left[\frac{\beta + \gamma z}{\beta + \gamma z_{\text{turn}}} \right] - \alpha(z - z_{\text{turn}}).$$

6. Discussion and conclusions. We have shown in § 4 how to solve equation (3.9), at least asymptotically, for arbitrary claims distribution and arbitrary $\beta(Z)$ for all Z for which $E_y E_{d\pi}\{\beta(Z)\} - \lambda\mu \geq 0$. We have also treated the case of linear $\beta(Z)$ and arbitrary claims distribution and solved asymptotically for $R(z)$ for all possible deterministic risk flows. For the case in which $E_y E_{d\pi}\{\beta(Z)\} - \lambda\mu < 0$, we introduced a second asymptotic solution. If the deterministic dynamics have a single rest point for $z > 0$, then the method developed in § 5.3 will provide a solution. If the deterministic dynamics have multiple rest points, then the solution using the error function will not be uniformly valid. Either we must patch together a number of error function solutions, or use a more complicated ansatz (see Mangel [9] for a discussion showing how to use the Airy and Pearcey functions).

In solving for the probability of survival for nonconstant premium dynamics, new collective ruin problems can be posed. For example, an optimal control problem can be formulated by looking to drive $R(z)$ to a target state by controlling $\beta(z)$. A solution of such a control problem could be used to determine an optimum premium policy or to compare different premium policies.

Many real insurance processes are characterized by seasonal or multiple claim intensities, so that λ is a vector. Our methods are easily adapted to that situation (Peters [11]). However, the problem of computing the time dependent probability of survival remains open.

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