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SEARCH FOR A RANDOMLY MOVING OBJECT*

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Abstract. After a brief discussion of the operational origin of search problems, the mathematical problem is formulated. The mathematical quantity of interest is the joint density for location of the object sought and unsuccessful search. When the object moves according to a diffusion process, this joint density satisfies a parabolic equation. After the introduction of scaled variables, the search equation can be approximately solved by the "ray method". The interpretation of the terms in the approximate solution is discussed. The case of constant diffusion and drift parameters and piecewise linear searching paths arises often in operational situations. This case is considered in detail.

1. Introduction. There are many operational instances in which a searcher seeks a moving object. For example, the tuna purse seine fishing fleet spends a considerable amount of time at sea searching for tuna schools [1]. The coast guards of maritime nations hold thousands of open ocean search and search and rescue missions each year. The search for naval vessels received considerable attention during World War II [2], and is still of interest. These operational problems are loosely characterized as follows. The object sought (called the target from now on) moves. The location of the target at the beginning of the search is not known precisely, but is described by a probability distribution. The search is characterized by a function that gives the probability of detecting the target, given the position of searcher and target. The mathematical aspects which are important to search planners can be divided into descriptive and optimal categories. In the descriptive problems, one wishes to characterize the joint density for target location and unsuccessful search. In the optimal problems, one wishes to choose a search plan that extremizes a given functional, e.g., the probability of detection [3].

Search planners are often constrained by computational limitations. Hand-held calculators and minicomputers are the most that an analyst can expect. Simplicity and speed of calculation are important factors when one is trying to implement a technique.

The traditional approach to search problems has been to convert from the search path to "search effort" [2], [3]. Search effort is typically measured in effective area searched or in time spent searching (in [2] and [3] the use of search effort is discussed in detail). The use of search effort may be warranted if the time available for the search is large, or if there are many searchers involved. In other cases, the use of search effort may not be appropriate.

In the last few years, a good deal of work has been done deriving necessary or necessary and sufficient conditions for optimal search tracks or optimal effort allocations (see, for example, [3], [4], [16], [17], [19]). Most of these results appeared as general, and sometimes very abstract, theorems. The more concrete problem of calculating the joint density for target location and unsuccessful search is still outstanding; this problem is treated in the present paper.

In this paper, it is assumed that the target moves as a diffusion process. For simplicity, the target is assumed to move in the plane and the searcher in space. The

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descriptive search problem is to find the joint density for target location and unsuccessful search. Let $(X(t), Y(t))$ be the position of the target at time t and let $S(t) = (S_x(t), S_y(t), S_z(t))$ be the position of the searcher at time t. The joint density of interest is denoted by $f(x, y, t, S)$, and is defined by

$$f(x, y, t, S)\,dx\,dy = \text{Prob}\,\{x \leqq X(t) \leqq x+dx,\ y \leqq Y(t) \leqq y+dy,\ \text{search along } S(\tau),\ 0 \leqq \tau \leqq t, \text{ was not successful}\}. \tag{1}$$

Once $f(x, y, t, S)$ is known, the conditional density, given unsuccessful search, $\rho(x, y, t|S)$, is

$$\rho(x, y, t|S) = \frac{f(x, y, t, S)}{\iint_D f(x, y, t, S)\,dx\,dy}. \tag{2}$$

In (2), D is the domain in which the target moves. The probability of detection by time t, P_t, is

$$P_t = 1 - \iint_D f(x, y, t, S)\,dx\,dy. \tag{3}$$

In this paper, the search for a moving target by a single searcher is studied. It is shown that the density $f(x, y, t, S)$ can be approximately calculated by asymptotically solving the equation that the density satisfies.

In § 2, the search equation for $f(x, y, t, S)$ is given and interpreted. This equation was derived by Hellman [5]. The components of the equation involve a covariance matrix representing diffusion, a drift vector and a detection function. It is possible that all coefficients depend on space and time. In the simplest case, the drift and covariance are constant, but the detection function is still a nonlinear function of space and time. In § 3, deterministic target motion is considered. In this case, the covariance matrix is identically zero. The search equation reduces to a first order equation, which is solved by the method of characteristics. The classical result [2] for a stationary target is obtained by specialization. In § 4 the search equation is put into nondimensional form. Since the use of nondimensional equations is not common in operations research, the scaling is discussed in detail. In § 5, the scaled search equation is solved approximately by using the "ray method" of J. B. Keller and co-workers [10]–[12]. The use of the ray method in search problems is new.

The ray solution shows clearly how the operational components of motion and search interact. The density $f(x, y, t, S)$ is asymptotically separable, in that it is a product of terms representing an initial value, motion and search. The case of constant drift vector and diffusion matrix and a piecewise linear searching path is of sufficient operational importance to deserve a separate discussion. In § 6, this case is considered.

It should be admitted from the outset that any "theory of search" is a simple abstraction of a complex operational system. A useful theory should show the interplay of the simple auxiliary models and the complex operational systems. Neither abstract mathematics nor purely numerical approaches provide information on this level. The approximate techniques introduced here do provide information on this level.

2. Search equation, target motion model, and detection model. Since the target moves as a diffusion process, $f(x, y, t, S)$ satisfies the following search equation [5]:

$$\frac{\partial f}{\partial t}=\frac{1}{2}\left[\frac{\partial^2}{\partial x^2}(a_{11}(x, y, t)f)+2\frac{\partial^2}{\partial x\,\partial y}(a_{12}(x, y, t)f)+\frac{\partial^2}{\partial y^2}(a_{22}(x, y, t)f)\right]$$
$$-\frac{\partial}{\partial x}(b_1(x, y, t)f)-\frac{\partial}{\partial y}(b_2(x, y, t)f)-\psi(x, y, t, S)f, \tag{4}$$

$$f(x, y, 0, S)=\rho_0(x, y). \tag{5}$$

Equations (4) and (5) contain information about the target motion model and detection model. The function $\rho_0(x, y)$ is the initial density for the location of the target:

$$\rho_0(x, y)\,dx\,dy=\text{Prob}\,\{x \leqq X(0) \leqq x+dx,\ y \leqq Y(0) \leqq y+dy\}. \tag{6}$$

The components of the right-hand side of (4) describe target motion and search. The matrix $(a_{ij}(x, y, t))$ is a measure of the stochastic or diffusive part of the target motion. It will be called the covariance matrix [6]. In many cases of practical interest, the covariance matrix is a constant matrix. In that case, it can be considered diagonal. In other cases, e.g., if the target is a lifeboat drifting in a rapid current, the covariance may depend upon position and time. It is possible that $(a_{ij}(x, t))$ is so small that it can be set equal to zero. The vector $(b_1(x, y, t), b_2(x, y, t))$ represents the deterministic or mean component of the target motion. It will be called the drift vector. There are many operational situations where the drift vector is constant. When both the drift and covariance are constant, the target motion will be called homogeneous. The case of homogeneous target motion is explicitly treated in § 6. There are operational situations in which the drift vector is not constant. For example, in the Straits of Florida off the eastern U.S. coast, the drift vector strongly depends on position. Another possibility is that the drift depends on a random variable. For example, in the *fleeing datum* problem ([2, p. 16], also see [7])

$$b_1(x, y, t)=c\cos\alpha, \qquad b_2(x, y, t)=c\sin\alpha.$$

Here, the target moves with known speed c but unknown bearing α; e.g., a uniform distribution for α may be assumed.

The detection function $\psi(x, y, t, S)$ is defined so that

$$\psi(x, y, t, S)\Delta t=\text{Prob}\,\{\text{detection in } (t, t+\Delta t)|X(t)=x,\ Y(t)=y,\ S(t)=s\}+o(\Delta t). \tag{7}$$

The exact form of the detection function depends on the physical detection apparatus. For airborne visual detection, a proposed model is [2]

$$\psi(x, y, t, S)=\frac{kS_z(t)}{[S_z(t)^2+(S_x(t)-x)^2+(S_y(t)-y)^2]^{3/2}}. \tag{8}$$

The parameter k in (8) depends upon the particular target type and environmental conditions. The U.S. Coast Guard has tables that can be used to find k as a function of prevailing environmental conditions. For surface search, a proposed model is [4]

$$\psi(x, y, t, S)=k\exp[-\beta\{(x-S_x(t))^2+(y-S_y(t))^2\}], \tag{9}$$

where k and β are parameters that depend upon the environmental conditions. A possible model for two-way radar search is

$$\psi(x, y, t, S) = -\frac{1}{S_t}\ln\left(1-\frac{1}{2}\operatorname{erfc}\left(a-\frac{b}{R^2}\right)\right), \tag{10}$$

where

$$R = [S_z(t)^2 + (S_x(t)-x)^2 + (S_y(t)-y)^2]^{1/2} \tag{11}$$

is the target to radar distance, S_t is the scan time of the radar, $\operatorname{erfc}(u)$ is the complementary error function and a, b, are parameters characterizing the radar. Equation (10) is derived by applying the statistical theory of detection to the radar equation (see [8], [8a]).

In actual airborne search operations, the path that the searcher uses is approximately piecewise linear. Two typical search patterns are illustrated in Fig. 1. When the searcher uses a piecewise linear path, the search will be called *linear search.* Linear search is considered in more detail in § 6.

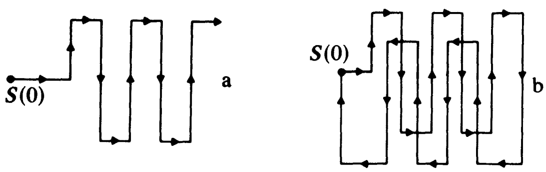

FIG. 1. *Two types of searching paths.* (a) *A path in which the end point is not fixed.* (b) *A path in which the end point is fixed.*

3. Deterministic target motion. When the covariance matrix is zero, the target moves deterministically. The search equation becomes

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(b_1(x, y, t)f) - \frac{\partial}{\partial y}(b_2(x, y, t)f) - \psi(x, y, t, S)f, \tag{12}$$

$$f(x, y, 0, S) = \rho_0(x, y). \tag{13}$$

Performing the differentiation in (12) gives

$$\frac{\partial f}{\partial t} + b_1\frac{\partial f}{\partial x} + b_2\frac{\partial f}{\partial y} = -(\psi(x, y, t, S) + \nabla \cdot b)f, \tag{14}$$

where

$$\nabla \cdot b = \frac{\partial b_1}{\partial x} + \frac{\partial b_2}{\partial y}.$$

Equation (14) can be solved by the method of characteristics [9]. Along the solution curves of

$$\begin{aligned} \frac{dx}{dt} &= b_1(x, y, t), \qquad x(0) = x_0, \\ \frac{dy}{dt} &= b_2(x, y, t), \qquad y(0) = y_0, \end{aligned} \tag{15}$$

(14) becomes

$$\frac{df}{dt} = -(\psi(x, y, t, S) + \nabla \cdot b)f, \qquad (16)$$
$$f(0) = \rho_0(x_0, y_0).$$

Denote the solutions of (15) by $(x(t, x_0, y_0)y(t, x_0, y_0))$. In the (x, y)-plane these solutions are a curve or ray that emanates from the point (x_0, y_0). The solution of (16) is then

$$\begin{aligned} &f(t, x(t, x_0, y_0), y(t, x_0, y_0)) \\ &\quad = \rho(x_0, y_0) \exp\left[-\int_0^t (\nabla \cdot b)\, d\tau\right] \exp\left[-\int_0^t \psi(x(\tau), y(\tau), \tau, S(\tau))\, d\tau\right]. \end{aligned} \qquad (17)$$

In (17) $x(\tau)$ and $y(\tau)$ are shorthand for $x(\tau, x_0, y_0)$ and $y(\tau, x_0, y_0)$.

Two special cases of (17) are of interest. If the deterministic flow is divergence free, so that $\nabla \cdot b$ vanishes, (17) becomes

$$f(t, x(t, x_0, y_0), y(t, x_0, y_0)) = \rho_0(x_0, y_0) \exp\left[-\int_0^t \psi(x(\tau), y(\tau), \tau, S(\tau))\, d\tau\right]. \qquad (18)$$

If the target does not move, so that $b = 0$, then $x(t) = x_0$, $y(t) = y_0$ and

$$f(t, x, y) = \rho_0(x, y) \exp\left[-\int_0^t \psi(x, y, \tau, S(\tau))\, d\tau\right]. \qquad (19)$$

Equation (19) was derived in [2] by a different method.

Note that (17) is "separable" in the sense that the density is a product of a term representing an initial value, a term representing target motion and a term representing search.

4. Scaled search equation. In the next section, it will be shown that the search equation (4) can also be solved in terms of rays and the associated ordinary differential equations. This ray method has been successfully used in a variety of other problems [10], [11], [12]. The approximate solution begins by introducing a small parameter ε, obtained by scaling the dimensional equation (4). Let T_c, L_c and a_c be characteristic values for time, length and covariance matrix. For instance, T_c could be the time available for the search, L_c^2 could be the variance of the initial density (or L_c some other reference length), and a_c could be the maximum value of $\|a\|$.

Let dimensionless variables $\hat{t}$, $\hat{x}$, $\hat{y}$, $\hat{b}$, $\hat{a}$, $\hat{\psi}$ and ε be defined by

$$\begin{aligned} &t = \hat{t}T_c, \quad x = \hat{x}L_c, \quad y = \hat{y}L_c, \\ &b_i(x, y, t) = \hat{b}_i(\hat{x}, \hat{y}, \hat{t})L_c/T_c, \\ &a_{ij}(x, y, t) = \hat{a}_{ij}(\hat{x}, \hat{y}, \hat{t})a_c, \\ &\varepsilon = a_cT_c/L_c^2, \qquad \psi(x, y, t, S) = \hat{\psi}(\hat{x}, \hat{y}, \hat{t}, S/L_c)/T_c. \end{aligned} \qquad (20)$$

It is assumed that $0 < \varepsilon \ll 1$. This will be true for many operational problems. It is also assumed that $\hat{b}_i$ and $\hat{\psi}$ are order 1, rather than order ε. The technique given below can also work when $\hat{b}_i$ and $\hat{\psi}$ are order ε.

In terms of the scaled variables, the search equation (4) becomes

$$\frac{\partial f}{\partial \hat{t}}=\frac{\varepsilon}{2}\left[\frac{\partial^2}{\partial \hat{x}^2}(\hat{a}_{11}f)+2\frac{\partial^2}{\partial \hat{x}\,\partial \hat{y}}(\hat{a}_{12}f)+\frac{\partial^2}{\partial \hat{y}^2}(\hat{a}_{22}f)\right] -\frac{\partial}{\partial \hat{x}}(\hat{b}_1 f)-\frac{\partial}{\partial \hat{y}}(\hat{b}_2 f)-\hat{\psi}f. \tag{21}$$

It is assumed that the initial density takes the form[1]

$$f(\hat{x},\hat{y},0,S)=\exp(-\Phi(\hat{x},\hat{y})/\varepsilon)\sum_{k=0}\varepsilon^k h_k(\hat{x},\hat{y}). \tag{22}$$

The scaling chosen here appears to be a natural one. If $\varepsilon=0$, then (21) reduces to the search equation in the case of deterministic target motion. Also, ε is the ratio of the diffusion coefficient of the target motion to the diffusion coefficient constructed from the characteristic parameters.

5. Ray solution of the search equation. The equations of interest are (21) and (22). They will be solved by the ray method; see [10], [11], [12]. Since the details of the method are presented in the references, some details may be eliminated here. In the sequel, the hats will be dropped.

A solution of equation (21) is sought in the form

$$f(x,y,t,S)=\exp(-\phi(x,y,t)/\varepsilon)\sum_{k=0}\varepsilon^k g_k(x,y,t,S). \tag{23}$$

In this equation $\phi(x, y, t)$ and the functions $g_k(x, y, t, S)$ are to be determined. The ansatz (23) will asymptotically, for small ε, satisfy the search equation if $\phi(x, y, t)$ and $g_0(x, y, t, S)$ satisfy the following equations [10], [11]:

$$\frac{\partial \phi}{\partial t}+b_1(x,y,t)\frac{\partial \phi}{\partial x}+b_2(x,y,t)\frac{\partial \phi}{\partial y}+\frac{a_{11}(x,y,t)}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 +a_{12}(x,y,t)\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)+\frac{a_{22}(x,y,t)}{2}\left(\frac{\partial \phi}{\partial y}\right)^2=0, \tag{24}$$

$$\frac{\partial g_0}{\partial t}+\left(b_1+a_{11}\frac{\partial \phi}{\partial x}+a_{12}\frac{\partial \phi}{\partial y}\right)\frac{\partial g_0}{\partial x}+\left(b_2+a_{12}\frac{\partial \phi}{\partial x}+a_{22}\frac{\partial \phi}{\partial y}\right)\frac{\partial g_0}{\partial y} =-\left(\psi(x,y,t,S)+\nabla\cdot b+\frac{a_{11}}{2}\frac{\partial^2 \phi}{\partial x^2}+a_{12}\frac{\partial^2 \phi}{\partial x\,\partial y}+\frac{a_{22}}{2}\frac{\partial^2 \phi}{\partial y^2}+\nabla\phi\cdot\nabla a\right)g_0. \tag{25}$$

In (25), $\nabla\phi\cdot\nabla a$ is shorthand notation for

$$\frac{\partial a_{11}}{\partial x}\frac{\partial \phi}{\partial x}+\frac{\partial a_{12}}{\partial y}\frac{\partial \phi}{\partial x}+\frac{\partial a_{12}}{\partial x}\frac{\partial \phi}{\partial y}+\frac{\partial a_{22}}{\partial y}\frac{\partial \phi}{\partial y}.$$

Equation (24) is solved by the method of characteristics. Set $p=\partial\phi/\partial x$, $q=\partial\phi/\partial y$, and define H by

$$H=b_1(x,y,t)p+b_2(x,y,t)q+\tfrac{1}{2}a_{11}(x,y,t)p^2+a_{12}(x,y,t)pq+\tfrac{1}{2}a_{22}(x,y,t)q^2. \tag{26}$$

[1] Any initial density can formally be put into the form (22) by setting $\Phi=-\varepsilon\ln\rho_0$, $h_0=1$ and $h_k=0$ for $k\geqq 1$. There are instances in which this procedure is not effective. In those cases, it is best to work with the adjoint equation to (21). Details are given in [13].

The characteristic equations are

$$
(27)\qquad
\begin{aligned}
&\frac{dx}{dt}=\frac{\partial H}{\partial p}, && x(0)=x_0,\\
&\frac{dy}{dt}=\frac{\partial H}{\partial q}, && y(0)=y_0,\\
&\frac{dp}{dt}=-\frac{\partial H}{\partial x}, && p(0)=\left.\frac{\partial \Phi}{\partial x}\right|_{(x_0,y_0)},\\
&\frac{dq}{dt}=-\frac{\partial H}{\partial y}, && q(0)=\left.\frac{\partial \Phi}{\partial y}\right|_{(x_0,y_0)},\\
&\frac{d\phi}{dt}=\frac{\partial \phi}{\partial t}+p\frac{dx}{dt}+q\frac{dy}{dt}, && \phi(0)=\Phi(x_0, y_0).
\end{aligned}
$$

The last equation can be written as

$$
(28)\qquad \frac{d\phi}{dt}=\tfrac{1}{2}[a_{11}(x, y, t)p^2+2a_{12}(x, y, t)pq+a_{22}(x, y, t)q^2].
$$

The solution of (27) gives a ray, $(x(t, x_0, y_0), y(t, x_0, y_0))$, in the plane. When this method is numerically implemented, a point (x, y) at time t is picked. The rays are inverted to find $x_0(t, x, y)$ and $y_0(t, x, y)$. This is the starting point of a ray that goes through (x, y) at time t. Equation (28) is then integrated from $(x_0(t, x, y), y_0(t, x, y))$. In this way, $\phi(x, y, t)=\phi(x_0(t, x, y), y_0(t, x, y))$ is constructed.

Along the rays generated by (27), (25) becomes

$$
(29)\qquad
\begin{aligned}
&\frac{dg_0}{dt}=-(\psi(x, y, t, S)+\Gamma(x, y, t))g_0,\\
&g_0(0)=h_0(x_0, y_0).
\end{aligned}
$$

In this equation

$$
(30)\qquad \Gamma(x, y, t)=\nabla\cdot b+\tfrac{1}{2}a_{11}\frac{\partial^2\phi}{\partial x^2}+a_{12}\frac{\partial^2\phi}{\partial x\,\partial y}+\tfrac{1}{2}a_{22}\frac{\partial^2\phi}{\partial y^2}+\nabla\phi\cdot\nabla a.
$$

The solution of (29),

$$
g_0(x, y, t, S)=g_0(x_0(x, y, t, S), y_0(x, y, t, S)),
$$

is obtained by a simple integration. Combining this result and (28) gives

$$
(31)\qquad
\begin{aligned}
f(x, y, t, S)=\ &h_0(x_0(x, y, t), y_0(x, y, t))\\
&\cdot\exp\left[-\frac{1}{\varepsilon}\phi(x_0(x, y, t), y_0(x, y, t))\right]\\
&\cdot\exp\left[-\int_0^t \Gamma(x(\tau, x_0, y_0), y(\tau, x_0, y_0), \tau)\,d\tau\right]\\
&\cdot\exp\left[-\int_0^t \psi(x(\tau, x_0, y_0), y(\tau, x_0, y_0), \tau, S(\tau))\,d\tau\right]+O(\varepsilon).
\end{aligned}
$$

The asymptotic form of $f(x, y, t, S)$ is thus a product of a term representing the initial value, two terms representing target motion, and a term representing search.

The ray procedure described here is the simplest one; it does not work in all instances (see, e.g., [10] or [12]). Some limitations will now be discussed.

If the domain D is bounded, then (31) must be modified to satisfy the boundary conditions. Such modifications are discussed in [10] for the general case and in [14] for the application to search problems. In addition, when there are boundaries present, there may be rays which are reflected from the boundaries. In general, the probability density at a given point is obtained by summing over all the rays which reach that point.

If the rays intersect or have an envelope (analogous to a caustic [15]), the simple procedure given here may break down at some points. In those cases, a more complicated ray ansatz is needed. The technique given here or some small modification, however, will be applicable to many problems of operational interest.

6. Homogeneous target motion and linear search. When the drift vector and covariance matrix do not depend on position, the motion is spatially homogeneous. When they are constant, the motion is homogeneous. The operational problem of homogeneous target motion and linear search is so common that it warrants a separate discussion. In addition, the ray equations simplify enough that they can be integrated by hand. First assume that the drift is spatially homogeneous, $b = (b_1(t), b_2(t))$, and that the covariance is constant, $a_{11} = \text{constant}$, $a_{12} = a_{21} = 0$, $a_{22} = \text{constant}$. The ray equations (27) become

$$
\begin{aligned}
&\frac{dx}{dt} = b_1(t) + a_{11}p, && x(0) = x_0,\\
&\frac{dy}{dt} = b_2(t) + a_{22}q, && y(0) = y_0,\\
&\frac{dp}{dt} = 0, && p(0) = \left.\frac{\partial \Phi}{\partial x}\right|_{(x_0, y_0)},\\
&\frac{dq}{dt} = 0, && q(0) = \left.\frac{\partial \Phi}{\partial y}\right|_{(x_0, y_0)},\\
&\frac{d\phi}{dt} = \tfrac{1}{2}[a_{11}p^2 + a_{22}q^2], && \phi(0) = \Phi(x_0, y_0).
\end{aligned}
\tag{32}
$$

Denote $\Phi_x = \partial\Phi/\partial x$, $\Phi_y = \partial\Phi/\partial y$. The solution of the ray equations is

$$
\begin{aligned}
x(t) &= x_0 + \int_0^t b_1(\tau)\, d\tau + a_{11}\Phi_x(x_0, y_0)t,\\
y(t) &= y_0 + \int_0^t b_2(\tau)\, d\tau + a_{22}\Phi_y(x_0, y_0)t,\\
\phi(t) &= \Phi(x_0, y_0) + \frac{t}{2}[a_{11}\Phi_x(x_0, y_0)^2 + a_{22}\Phi_y(x_0, y_0)^2].
\end{aligned}
\tag{33}
$$

In order to find $x_0(x, y, t)$ and $y_0(x, y, t)$ one needs to solve the simultaneous equations

$$x_0 + a_{11}\Phi_x(x_0, y_0)t = x(t) - \int_0^t b_1(\tau)\, d\tau, \tag{34}$$

$$y_0 + a_{22}\Phi_y(x_0, y_0)t = y(t) - \int_0^t b_2(\tau)\, d\tau. \tag{35}$$

Let us assume that the solution $(x_0(x, y, t), y_0(x, y, t))$ is known. Then,

$$\phi(x, y, t) = \Phi(x_0(x, y, t), y_0(x, y, t)) \tag{36}$$
$$+\frac{1}{2}\left[\frac{(x - \int_0^t b_1(\tau)\, d\tau - x_0(x, y, t))^2}{a_{11}t} + \frac{(y - \int_0^t b_2(\tau)\, d\tau - y_0(x, y, t))^2}{a_{22}t}\right].$$

The function $f(x, y, t, S)$ has a Gaussian component. That this is true can be seen by going back to (21), setting $\psi = 0$, Fourier transforming and calculating the Green's function. In order to calculate $g_0(x, y, t, S)$ two integrals are needed. The first integral involves $\Gamma(x, y, t)$. In this case

$$\Gamma(x, y, t) = \frac{a_{11}}{2}\frac{\partial^2\phi}{\partial x^2} + \frac{a_{22}}{2}\frac{\partial^2\phi}{\partial y^2}. \tag{37}$$

In this case, it is easy to explicitly find $\Gamma(x, y, t)$ by using the chain rule. An alternate procedure, described in [10] and [11], uses the Jacobian of the transformation from ray to physical space to simplify the equation for $g_0(x, y, t, S)$.

As a particularly simple example, let us assume that $\Phi(x, y)$ has the special form

$$\Phi(x, y) = \frac{\alpha}{2}x^2 + \frac{\beta}{2}y^2. \tag{38}$$

Equation (37) simplifies to

$$\Gamma(x, y, t) = \frac{\alpha}{1 + \alpha t} + \frac{\beta}{1 + \beta t}. \tag{39}$$

The second integral needed to determine $g_0(x, y, t, S)$ is the integral of the detection function. Partition the interval $(0, t)$ into pieces $(t_0 = 0, t_1)$, $(t_1, t_2), \cdots,$ $(t_{n-1}, t_n = t)$, and assume that the search path is piecewise linear (which was called *linear search* earlier). The position of the searcher in the interval (t_{j-1}, t_j) is assumed to be

$$\begin{aligned} S_x(t) &= S_x(t_{j-1}) + v_{xj}(t - t_{j-1}),\\ S_y(t) &= S_y(t_{j-1}) + v_{yj}(t - t_{j-1}),\\ S_z(t) &= S_z(t_{j-1}) + v_{zj}(t - t_{j-1}). \end{aligned} \tag{40}$$

In this equation, $S(t_{j-1})$ is the position of the searcher at time t_{j-1} and (v_{xj}, v_{yj}, v_{zj}) is the velocity of the searcher. It is assumed to be constant for $t_{j-1} \leqq t \leqq t_j$.

The integral of the detection function becomes

$$\int_0^t \psi(x(\tau), y(\tau), \tau, S(\tau))\, d\tau = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \psi(x(\tau), y(\tau), \tau, S(\tau))\, d\tau. \tag{41}$$

To show how this formula can be used, assume that the searcher is trying to visually detect the target. Then $\psi(x, y, t, S)$ is given by (8). Also assume that the target motion is homogeneous, so that b_1 and b_2 are constant. Finally, assume that the altitude of the searcher is constant, so that $S_z(t) = h$, say. This last assumption does not cause any loss of generality, but it does help compact the results that follow. With these assumptions, the integral of the detection function is

$$
\begin{aligned}
(42)\qquad \int_{t_{j-1}}^{t_j} \psi(x(\tau), y(\tau), \tau, S)\, d\tau
&= \int_{t_{j-1}}^{t_j} kh[h^2 + \{S_x(t_{j-1}) + v_{xj}(\tau - t_{j-1}) - x_0 - b_1(\tau - t_{j-1}) \\
&\qquad - a_{11}\Phi_x(x_0, y_0)(\tau - t_{j-1})\}^2 \\
&\qquad + \{S_y(t_{j-1}) + v_{yj}(\tau - t_{j-1}) - y_0 - b_2(\tau - t_{j-1}) \\
&\qquad - a_{22}\Phi_y(x_0, y_0)(\tau - t_{j-1})\}^2]^{-3/2}\, d\tau.
\end{aligned}
$$

Define A_j, B_j, and C_j by

$$
\begin{aligned}
&A_j = h^2 + [S_x(t_{j-1}) - x_0(x, y, t_j)]^2 + [S_y(t_{j-1}) - y_0(x, y, t_j)]^2, \\
(43)\qquad &B_j = 2[S_y(t_{j-1}) - y_0(x, y, t_j)][v_{yj} - b_2 - a_{22}\Phi_y(x_0, y_0)] \\
&\qquad + 2[S_x(t_{j-1}) - x_0(x, y, t_j)][v_{xj} - b_1 - a_{11}\Phi_x(x_0, y_0)], \\
&C_j = (v_{xj} - b_1 - a_{11}\Phi_x(x_0, y_0))^2 + (v_{yj} - b_2 - a_{22}\Phi_y(x_0, y_0))^2.
\end{aligned}
$$

The integral of the detection function becomes

$$
\begin{aligned}
(44, 45)\qquad \int_{t_{j-1}}^{t_j} \psi(x(\tau), y(\tau), \tau, S(\tau))\, d\tau
&= \int_{t_{j-1}}^{t_j} \frac{kh\, d\tau}{[A_j + B_j(\tau - t_{j-1}) + C_j(\tau - t_{j-1})^2]^{3/2}} \\
&= kh\left[\frac{2(2C_j(t_j - t_{j-1}) + B_j)}{Q_j(A_j + B_j(t_j - t_{j-1}) + C_j(t_j - t_{j-1})^2)^{1/2}} - \frac{2B_j}{Q_j A_j^{1/2}}\right].
\end{aligned}
$$

In (45), $Q_j = 4A_jC_j - B_j^2$. It is easy to show that $Q_j > 0$ as long as the searcher is moving. The functions A_j, B_j and C_j depend on $x_0(x, y, t_j)$, $y_0(x, y, t_j)$. Thus, (34) and (35) need to be solved for the evaluation of the integral of the detection function. These results can be combined with (31) to give

$$
\begin{aligned}
(46)\qquad f(x, y, t, S) &\approx h_0(x_0(x, y, t), y_0(x, y, t)) \\
&\cdot \exp\left[-\frac{1}{\varepsilon}\Phi(x_0(x, y, t), y_0(x, y, t)) - \frac{1}{2\varepsilon}\frac{(x - b_1 t - x_0(x, y, t))^2}{a_{11}t} - \frac{1}{2\varepsilon}\frac{(y - b_2 t - y_0(x, y, t))^2}{a_{22}t}\right] \\
&\cdot \exp\left[-\int_0^t \Gamma(x(\tau), y(\tau), \tau)\, d\tau\right] \\
&\cdot \exp\left[-\sum_{j=1}^{n} kh\left(\frac{2(2C_j(t_j - t_{j-1}) + B_j)}{Q_j(A_j + B_j(t_j - t_{j-1}) + C_j(t_j - t_{j-1})^2)^{1/2}} - \frac{2B_j}{Q_j A_j^{1/2}}\right)\right].
\end{aligned}
$$

Equation (46) is a general result that holds for homogeneous target motion and linear search with visual detection. In order to use (46), the initial density must be specified. Then $h_0(x_0, y_0)$ and $\Phi(x_0, y_0)$ are known. Next, one evaluates $(x_0(x, y, t), y_0(x, y, t))$ by using (34) and (35). Then $\phi(x, y, t)$ is calculated and integrated. Next, $g_0(x, y, t, S)$ is evaluated. To do this, two pieces are needed. The first is $\Gamma(x, y, t)$ and the second is knowledge of A_j, B_j, and C_j. When these are known, $g_0(x, y, t, S)$ can be found.

7. Discussion. The technique introduced here for the solution of the search equation can be easily implemented numerically, since only ordinary differential equations are involved. In fact, it is possible to calculate the density $f(x, y, t, S)$, in a problem of operational interest, using an HP-67 programmable calculator. When the target motion is homogeneous and the search is linear, it is possible to obtain exact solutions of the ray equations.

In many cases the drift vector is not known with complete certainty, but has a certain distribution. Such motion is called conditionally deterministic [3]. The technique developed here can be applied to these problems too. The search equation is solved for $f_c(x, y, t, S)$ conditioned on the drift vector. Then $f_c(x, y, t, S)$ is integrated against the density of the drift vector.

In [17], Lukka derives necessary conditions for the search track that maximizes the probability of detecting the target by a fixed time T. In addition to having to find $f(x, y, t, S)$, one needs the solution of the equation adjoint to (4), and then the solution of a two-point boundary value problem. The ray method can also be used to solve the adjoint equation [13], and to simplify the integrals arising in the boundary value problem. Thus the work presented here represents part of the solution of the optimal track problem.

An alternate procedure for optimal search track calculation is to work directly with the probability of detection given by (3), and to find the optimal track by some sort of nonlinear programming procedure. In [18], this program was attempted, but was hampered by a limited ability to calculate the density $f(x, y, t, S)$. The method introduced here reduces this limitation.

Previous work on moving target search problems has been quite theoretical or numerical. There has been little use of approximate methods such as the ray techniques introduced here. Thus, the present work complements previous work. The approximate methods have an advantage in that one can see more clearly how the various components of the problem interact.

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