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SMALL FLUCTUATIONS IN SYSTEMS WITH MULTIPLE LIMIT CYCLES*

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Abstract. We consider the effects of small random perturbations on deterministic systems of differential equations. The deterministic systems of interest have multiple limit cycles and may undergo a bifurcation (the Hopf bifurcation). We formulate a first exit problem for experiments beginning near stable and unstable limit cycles. The unstable limit cycle is surrounded by an annulus. Of interest is the probability of first exit from the annulus through a specified boundary, conditioned on initial position. The diffusion approximation is used, so that the conditional probability satisfies a backward diffusion equation. Approximate solutions of the backward equation are constructed by an asymptotic method. The behavior of the stochastic system in the vicinity of stable and unstable limit cycles is compared. When the deterministic system exhibits the Hopf bifurcation, the above analysis must be modified. Uniform solutions of the backward equation are constructed. Numerical examples are used to compare the theory with Monte Carlo experiments.

Introduction. In recent years, the analysis of oscillatory nonlinear chemical and biological systems has received considerable attention. Often the systems of interest can be modeled by a deterministic differential equation of the form

$$\dot{x} = b(x, \mu), \qquad x(0) = x_0 \tag{1.1}$$

where $x = (x^1, x^2)$ and μ is a parameter. In some cases, (1.1) may have multiple periodic solutions, i.e., multiple limit cycles. In a typical case (Fig. 1b), the system (1.1) could have a steady state P that is a focus. The steady state could be surrounded by an unstable limit cycle U; the unstable limit cycle surrouned by a stable limit cycle L. In this case the system exhibits "threshold" behavior. If the initial phase point x_0 is in the region contained by U, then the solution of (1.1) will exhibit damped oscillations and will approach P. If x_0 is in the region bounded by U and L, then the solution of (1.1) will exhibit undamped oscillations and will approach L. Phase points initially on U remain there indefinitely, in the absence of fluctuations. The other source of threshold behavior in dynamical systems is multiple steady states. The effects of fluctuations on such systems was studied by Mangel (1979).

For example, the following equations have been proposed as a model of the lac-operon (Sanglier and Nicolis (1976)):

$$\dot{x}^1 = -2k_8(x^1)^2 + \frac{2k_8'(k_8(x^1)^2+\tau)}{(k_3(x^2)^2+k_1'+k_2')} + \frac{\alpha x^2(k_3(x^2)^2+k_1'+k_2')}{D(x^1, x^2; k_1'+k_2')}, \tag{1.2}$$

$$\dot{x}^2 = -2k_3(x^2)^2\left(\frac{(k_8(x^1)^2+\tau)}{(k_3(x^2)^2+k_1'+k_2')}\right) + k_4 + \frac{[x_0^2 - k_6x^2]\alpha(k_3(x^2)^2+k_1'+k_2')}{D(x^1x^2; k_1'+k_2')} \tag{1.3}$$

where k_8, k_8', τ, α, k_3, k_4, k_6, k_2', and x_0^2 are constants given by Sanglier and Nicolis (1976), k_1' is a parameter and

$$D(x^1, x^2; k_1'+k_2') = k_7\{k_2(k_8(x^1)^2+\tau) + k_2'(k_3(x^2)^2+k_1'+k_2')\}. \tag{1.4}$$

According to Sanglier and Nicolis (1976), as k_1' varies, the system (1.2), (1.3) exhibits the following behavior:

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a. For $k_1' > .1$, the system (1.2, 1.3) has a single steady state P that is a focus. At $k_1' = .1$ the system undergoes a bifurcation.

b. For $.000248 < k_1' < .1$, the stable focus P is surrounded by an unstable limit cycle, which is surrounded by a stable limit cycle. At $k_1' = .000248$, the system undergoes another bifurcation.

c. For $k_1' < .000248$, the system has multiple steady states.

Other examples of chemical systems with multiple limit cycles are found in Cohen (1972) and Burns, Bailey and Luss (1973). Examples of biological systems with multiple limit cycles are found in Bazekin (1975).

Equation (1.1) is completely deterministic. The fluctuations inherent to all natural systems have been ignored. In this paper, we study the effects of fluctuations on systems with single and multiple limit cycles. Three types of periodic behavior are of interest here. They are: (1) A fixed, stable limit cycle, surrounding an unstable focus (Fig. 1a). (2) A fixed unstable limit cycle, surrounding a stable focus and enclosed by a fixed, stable limit cycle (Fig. 1b). (3) The Hopf bifurcation problems: the deterministic dynamics depend upon a parameter μ. As $\mu \downarrow 0$, a stable limit cycle coalesces with an

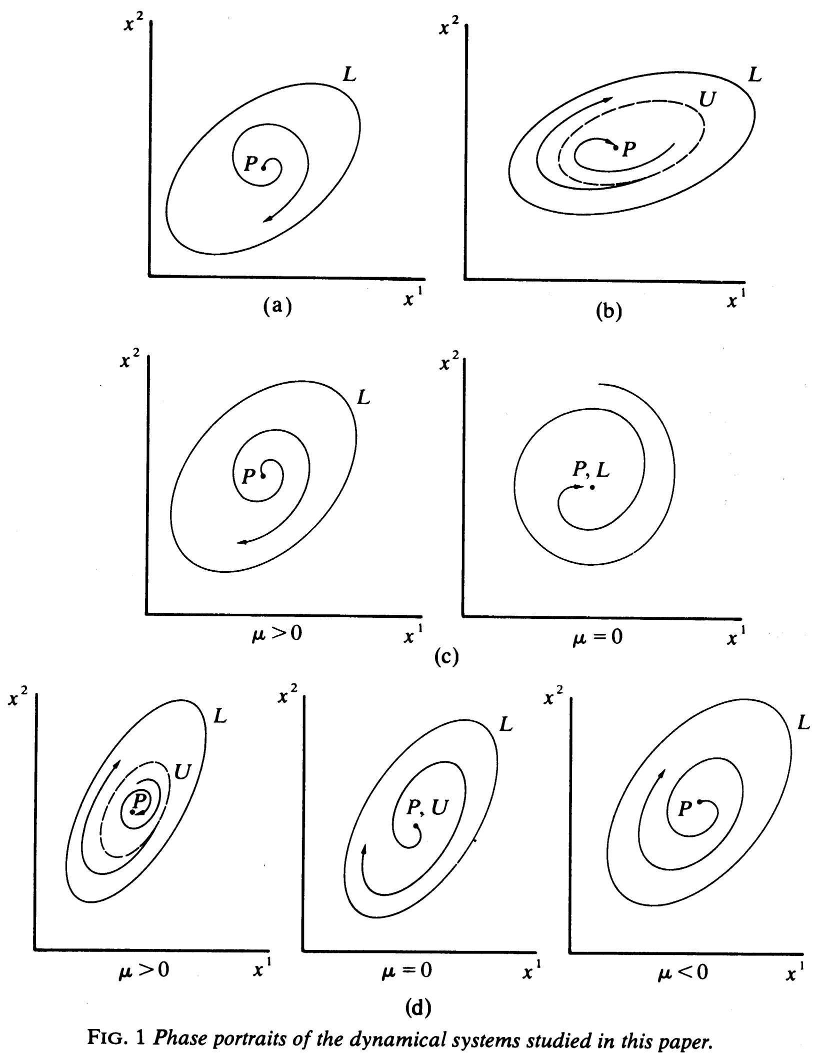

FIG. 1 *Phase portraits of the dynamical systems studied in this paper.*

unstable focus (at $\mu = 0$). The limit cycle disappears and the focus becomes stable (Fig. 1c). A "dual" bifurcation, in which an unstable cycle and stable focus coalesce, is shown in Fig. 1d.

The stable limit cycle, with superposed fluctuations, was treated by Ludwig (1975), White (1977), and Van Kampen (1976). It is included here for two reasons. First, our treatment is slightly different from the others. Second, the treatment given here leads to an interesting comparison of the stochastic dynamics of stable and unstable limit cycles. The unstable limit cycle contained by a stable limit cycle arises in chemical dynamics. In the engineering literature, bifurcations involving unstable limit cycles are sometimes called "hard oscillations." The Hopf bifurcation, and dual Hopf bifurcation, arise in many situations (Marsden and McCracken (1976)). Our interest is again motivated by chemical reaction dynamics (Haken (1978); Cohen (1972); Uppal et al., 1974).

When fluctuations are superposed upon the deterministic dynamics (1.1), a number of questions can be posed. The type of question that should be posed depends upon the nature of the deterministic dynamics. First, consider the case of a stable limit cycle (Fig. 1a). Since the deterministic attraction is always towards L, the question of most interest involves how fluctuations may drive the system away from the limit cycle. Let $\tilde{x}(t)$ denote the random variable obtained by superposing fluctuations on (1.1). (See § 2.) In this case, $x(t)$ in (1.1) should be an appropriate conditional average of $\tilde{x}(t)$ (Mangel, 1979). Let:

$$v(t, x)\,dx = Pr\{x \leqq \tilde{x}(t) \leqq x + dx\}. \tag{1.5}$$

Thus, $v(t, x)$ is the probability density for $\tilde{x}(t)$. It is a natural choice of a function that describes the stochastic dynamical system obtained when fluctuations are superposed. If we let $t \to \infty$, then $v(t, x) \to v(x)$, the equilibrium or stationary density, which gives the probability of eventually finding the process in the interval $(x, x + dx)$. Associated with $v(t, x)$ is an initial density, $v_0(\tilde{x}(0))$, that characterizes the distribution of the random variable $\tilde{x}(0) = x_0$.

The initial phase point is crucial when considering an unstable limit cycle U (Fig. 2). A phase point initially in the vicinity of U leaves any neighborhood of U with probability 1. Even if $\tilde{x}(0) \in U$, fluctuations will drive the phase point away from U. Ideally one would like to calculate the probability that, given $\tilde{x}(0) = x$, the phase point reaches a neighborhood of the node P rather than a neighborhood of the stable limit cycle L. In general, this problem is too difficult to solve. An alternative problem is the following one. Let s measure distance normal to the unstable limit cycle, chosen so that $s > 0$ corresponds to an outward orientation.

Consider two contours:

$$S_1 = \{x : s(x) = s_1\}, \qquad S_2 = \{x : s(x) = s_2\} \tag{1.6}$$

(see Fig. 2), with $s_1 > 0$, $s_2 < 0$. Consider the probability

$$u(t, x) = Pr\{\text{by time } t,\ \tilde{x}(t) \text{ has left the annulus } (S_1, S_2) \text{ through } S_1 | \tilde{x}(0) = x\}. \tag{1.7}$$

This probability is a function of initial position. The stationary version of (1.7) is $u(x)$, which is the probability that $\tilde{x}(t)$ first exits from (S_1, S_2) through S_1.

For the case of a system exhibiting the Hopf bifurcation, (Fig. 1c), we are again interested in the density for $\tilde{x}(t)$. Now the density is $v(t, x; \mu)$, where μ is the parameter characterizing the deterministic bifurcation. Consider now the dual Hopf system (Fig. 1d). For small μ, a phase point will leave a neighborhood of P or U and approach L with

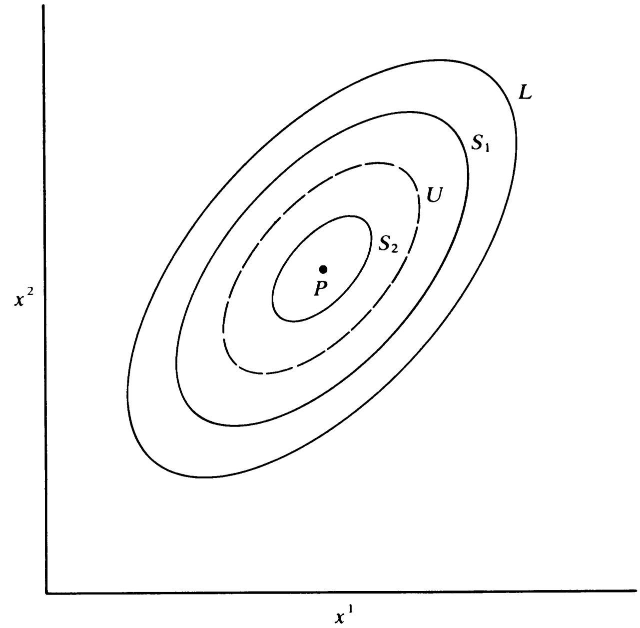

FIG. 2. *The first exit problem for the unstable limit cycle.*

probability 1. The singularity at P/U for $\mu = 0$ will be evidenced by very slow deterministic repulsion from P. Let $\bar{L}$ be a neighborhood of the stable limit cycle and let

$$T(x) = E\{t : \tilde{x}(t) \in \bar{L}, \tilde{x}(s) \notin \bar{L}, s < t | \tilde{x}(0) = x, \tilde{x}(t) \text{ reaches } \bar{L}\}. \tag{1.8}$$

Thus, $T(x)$ is the expected time to reach $\bar{L}$, given that $\tilde{x}(0) = x$.

In order to calculate the above quantities, one first needs to introduce a stochastic kinetic equation. This is done in § 2. First, we specify the deterministic dynamical equations corresponding to the systems with multiple limit cycles. Second, we modify these equations by the addition of a random function. This is the Langevin approach. We obtain a stochastic kinetic equation that is, usually, too difficult to treat directly. We treat the kinetic equations by the diffusion approximation of Papanicolaou and Kohler (1974). In this approximation, $v(t, x)$, $u(t, x)$ and $T(x)$ all satisfy deterministic partial differential equations. A small parameter, characterizing the intensity of the fluctuations, arises in derivation of the stochastic equations.

In § 3, we analyze canonical problems corresponding to stable and unstable limit cycles and the Hopf bifurcation problems. Various incomplete special functions arise in the analysis of these canonical problems. These functions are generalized in § 4, where we calculate $v(t, x)$ and $u(t, x)$ by the use of formal asymptotic methods, for stable and unstable (fixed) limit cycles. The stationary solutions $v(x)$, $u(x)$ have interesting interpretations. In § 5, we construct $v(t, x; \mu)$ and $u(t, x; \mu)$ for Hopf-type deterministic dynamical systems. We show that the solutions in § 4 break down and show how uniformly valid solutions can be obtained. In § 6, we present numerical examples that illustrate some of the phenomena discussed in §§ 4–5.

2. Deterministic and stochastic kinetic equations. First, we characterize the deterministic equations that lead to the phase portraits of interest. Second, we formulate the stochastic kinetic equations and the diffusion approximation.

2.1. Deterministic kinetic equations. We assume that

$$\dot{x}^i = b^i(x, \mu), \qquad x \in R^2, \mu \in R^1, \tag{2.1}$$

has a periodic solution Φ. Let P be a steady state of (2.1): $b^i(P, \mu) = 0$ for all i. Introduce new coordinates: s, which measures distance normal to Φ (with positive outward orientation), and θ which measures distance along Φ. (If Φ were a circle, then s is the radial variable and θ the angular variable.) From (2.1), we obtain equations for s and θ, of the form

$$\begin{aligned} \dot{s} &= \tilde{b}^s(s, \theta, \mu), \\ \dot{\theta} &= \tilde{b}^\theta(s, \theta, \mu). \end{aligned} \tag{2.2}$$

The variable θ is assumed to be periodic, with known period Θ, and increases in a counter-clockwise sense. The limit cycle is stable if

$$\frac{\partial}{\partial s}(\tilde{b}^s(0, \theta, \mu)) < 0 \tag{2.3}$$

and is unstable if

$$\frac{\partial}{\partial s}(\tilde{b}^s(0, \theta, \mu)) > 0. \tag{2.4}$$

If $\partial/\partial s(\tilde{b}^s(0, \theta, \mu)) = 0$, then the limit cycle is neutrally stable. This phenomenon occurs at the Hopf bifurcation point.

To consider the Hopf bifurcation, we return to (2.1). Let $B = (b^i_{,j}(P, \mu))$, and let $\lambda(\mu)$, $\lambda^*(\mu)$ denote the eigenvalues of B. The Hopf bifurcation is characterized by the following properties:

(1) When $\mu < 0$, $\lambda(\mu) \neq \lambda^*(\mu)$ are located in the left-half plane.
(2) When $\mu = 0$, the eigenvalues are located on the imaginary axis.
(3) When $\mu > 0$, the eigenvalues are located in the right-half plane. Also, the following condition is satisfied:

$$\frac{d}{d\mu} \operatorname{Re} \lambda(\mu)|_{\mu=0} \equiv \gamma_1 \neq 0. \tag{2.5}$$

There are analogous (dual) conditions for the dual Hopf bifurcation. Let

$$z = r e^{i\phi} = x^1 + ix^2. \tag{2.6}$$

Fenichel (1975) (see also Arnold, 1972) has shown that (2.1) can be put into the canonical form (for small μ)

$$\begin{aligned} \dot{r} &= \pm(b_1 r^3 - \eta\gamma_1 r) \equiv b^r(r, \phi, \eta), \\ \dot{\phi} &= \lambda_2 + b_2 r^2 + \eta\gamma_2 r \equiv b^\phi(r, \phi, \eta) \end{aligned} \tag{2.7}$$

where $r = r(s, \theta)$, $\phi = \phi(s, \theta)$ are regular functions, γ_1 is defined in (2.5), $\lambda_2 > 0$ and $b_1, b_2 \neq 0$. The ($\pm$) sign in (2.7) is included so that both the Hopf bifurcation and dual Hopf bifurcation can be treated. The function $\eta(\mu)$ is a regular function of the parameter and

$$\eta(0) = 0. \tag{2.8}$$

At the bifurcation point, $\eta=0$, note that

(2.9)
$$b^r(0,\phi,0)=b^r_{,r}(0,\phi,0)=b^r_{,rr}(0,\phi,0)=0,$$
$$b^r_{,rrr}(0,\phi,0)\neq 0.$$

These conditions will be used later.

2.2 Stochastic kinetic equation and diffusion approximation. Equation (2.1) is approximate in that it completely ignores fluctuations. On the other hand, deterministic equations (e.g., the "law of mass action" in chemistry) often yield correct predictions. Such deterministic equations are successful for two reasons. First, the fluctuations are of small intensity. Second, the fluctuations occur on a time scale rapid compared to the macroscopic equation. Accordingly, to obtain a stochastic kinetic equation we replace (2.1) by a Langevin-like equation for a random variable $\tilde{x}_\alpha(t)$:

(2.10)
$$\frac{d\tilde{x}^i_\alpha}{dt}=b^i(\tilde{x}_\alpha,\mu)+\sqrt{\frac{\varepsilon}{\alpha^2}}f^i_j(\tilde{x}_\alpha)Y^j(t/\alpha^2).$$

In (2.10), $Y(s)$ is a stationary, zero mean process, satisfying the mixing condition of Papanicolaou and Kohler (1974). The parameter $\varepsilon, 0<\varepsilon\ll 1$, characterizes the intensity of the fluctuations. In chemical systems, for instance,

(2.11)
$$\varepsilon=V_e/V$$

where V is the volume of the reacting system and V_e is the elementary volume, i.e., the volume of a sub-unit of the reacting system (Kubo et al. (1973), Van Kampen (1976)). The parameter $\alpha, 0\leqq\alpha\ll 1$ characterizes the time scale of the fluctuations. As $\alpha\to 0$, $\tilde{x}_\alpha(t)$ converges to a diffusion process (Papanicolaou and Kohler, 1974). The density $v(t,x)$, defined in § 1, satisfies the following differential equation (Papanicolaou and Kohler (1974)):

(2.12)
$$v_t=\frac{\varepsilon}{2}(a^{ij}v)_{ij}-((b^i+\varepsilon c^i)v)_i.$$

The probability $u(t,x)$ satisfies

(2.13)
$$u_t=\frac{\varepsilon}{2}a^{ij}u_{ij}+(b^i+\varepsilon c^i)u_i.$$

The expected time, $T(x)$, satisfies

(2.14)
$$-1=\frac{\varepsilon}{2}a^{ij}T_{ij}+(b^i+\varepsilon c^i)T_i.$$

In (2.12)–(2.14), subscripts indicate partial derivatives and repeated indices are summed from 1 to n. Also,

(2.15)
$$a^{ij}(x)=f^i_kf^j_l(\gamma^{kl}+\gamma^{lk}),$$
$$c^i(x)=\gamma^{kl}f^j_k\frac{\partial}{\partial x^j}f^i_l$$

where

$$\gamma^{kl}=\int_0^\infty E[Y^k(s)Y^l(0)]\,ds.$$

In later sections, we will specify the appropriate boundary conditions for (2.12)–(2.13). Equation (2.14) is treated elsewhere (Mangel (1979)).

3. Canonical problems and special functions. In this section, one dimensional canonical versions of (2.12, 2.13) are studied. The results will be generalized in the next section. The stationary versions of (2.12, 2.13) and appropriate boundary conditions are

$$0=\frac{\varepsilon}{2}(av)_{xx}-(bv)_x;\qquad \int_{-\infty}^{\infty} v(x)\,dx=1;\ v\to 0 \text{ as } |x|\to\infty, \tag{3.1}$$

$$0=\frac{\varepsilon}{2}au_{xx}+bu_x;\qquad u(x_1)=0,\qquad u(x_2)=1,\quad x_1<x_2. \tag{3.2}$$

We assume that the deterministic system

$$\dot{x}=b(x,\mu) \tag{3.3}$$

has a steady state at $x=x_0$. The deterministic equation (3.3) corresponds to the following degenerate "planar" dynamical system. Let (r,θ) denote the usual polar coordinates and consider

$$\begin{aligned}\dot{r}&=b(r,\theta;\mu),\\ \dot{\theta}&=0.\end{aligned} \tag{3.3a}$$

If we further require that there are no fluctuations in θ, then the system in (3.3a) formally reduces to the one-dimensional equation (3.3). This formal reduction requires one caveat. For a truly two-dimensional system in polar coordinates, the normalization condition given in (3.1) would not be valid. Instead, a condition of the following form would be required

$$\iint v(r,\theta)\,dr\,d\theta=1. \tag{3.1a}$$

The canonical system (3.1) is studied here to motivate a solution of the more complex problem.

Let $b'(x,\mu)$ denote the derivative of $b(x,\mu)$ with respect to x. It is assumed that when $\mu\neq 0$, $|b'(x_0,\mu)|>0$ and that when $\mu=0$, $b'(x_0,\mu)=0$. If $b'(x_0,\mu)<0$, the steady state is stable and equation (3.1) is of interest. If $b'(x_0,\mu)>0$, the steady state is unstable and equation (3.2) is of interest. For the rest of this section, it will be assumed that the diffusion coefficient a in (3.1, 2) is constant. The solutions of (3.1) and (3.2) are

$$v(x)=k\exp\left[\int^x \frac{2b}{\varepsilon a}\,dz\right]; \tag{3.4}$$

$$u(x)=k'\int_{x_1}^{x}\exp\left[-\int^y \frac{2b}{\varepsilon a}\,dz\right]dy; \tag{3.5}$$

The constants k, k' are determined by the integrability conditions and boundary conditions. For small ε, we use Laplace's method (Olver, 1974) to simplify (3.4)–(3.5). We obtain, for μ bounded way from zero,

$$v(x)=k\exp\left[\frac{-|b'(x_0)|(x-x_0)^2}{\varepsilon a}\right]+O(\sqrt{\varepsilon}), \tag{3.6}$$

$$u(x)=k'\int_{x_1}^{x}\exp\left[\frac{-b'(x_0)(y-x_0)^2}{\varepsilon a}\right]dy+O(\sqrt{\varepsilon}). \tag{3.7}$$

The integral appearing in (3.7) is the error integral

$$E(z)=\int_{z_0}^{z} e^{-s^2/2}\,ds. \tag{3.8}$$

The correction term in (3.7) involves the derivative of the error integral $E'(z)=e^{-z^2/2}$. The error integral satisfies the following differential equation:

$$\frac{d^2E}{dz^2}=-z\frac{dE}{dz}, \qquad -\infty \leqq z_0 \leqq z \leqq z_1 \leqq \infty. \tag{3.9}$$

The results in (3.6) and (3.7) correspond to locally Gaussian densities. The appearance of such Gaussian densities is due to the linearization process involved in Laplace's method. The use of a simple linearization requires that $|b'(x_0,\mu)|\neq 0$.

There are, however, instances in which a simple linearization cannot be used. For a two-dimensional system, this would be true at the Hopf bifurcation point. In this section an analogue of the Hopf bifurcation will be studied. Assume that when $\mu=0$, the following conditions hold:

$$\begin{aligned} &b'(x_0,0)=b''(x_0,0)=0,\\ &b'''(x_0,0)\neq 0. \end{aligned} \tag{3.10}$$

Hence, when applying Laplace's method, for μ near 0, we must use four terms in the Taylor expansion of $\int^x b/\varepsilon a\,ds$.

Instead of the error integral (also see Mangel, (1977, 1979)) where analogous problems for multiple steady states are treated), one finds that

$$u(x)=k'\int_{x_1}^{x}\exp\left[-\left(\frac{b'''(x_0,\mu)(y-x_0)^4}{12\varepsilon a}+\frac{b''(x_0,\mu)(y-x_0)^3}{3\varepsilon a}+\frac{b'(x_0,\mu)(y-x_0)^2}{\varepsilon a}\right)\right]dy+O(\varepsilon^{3/4}). \tag{3.11}$$

By a change of variables, we obtain that

$$u(x)\sim c\int_{\tilde{x}_1(x_1)}^{\tilde{x}(x)}\exp\left[-\frac{y^4}{4}+\frac{\eta(\mu)y^2}{2}\right]dy \tag{3.12}$$

where $\tilde{x}_1(x_1)$, $\tilde{x}(x)$ and $\eta(\mu)$ are regular functions of their arguments and $\eta(0)=0$. The result (3.12) can be obtained by applying Levinson's theorem (Levinson, 1962) directly to (3.5). Similarly, we find for the density $v(x)$:

$$v(x)\sim c'\exp\left(\frac{-y^4}{4}+\frac{\eta(\mu)}{2}y^2\right). \tag{3.13}$$

Thus, we are led to a special function, the incomplete Hopf integral,

$$H_\pm(z,\beta)\equiv\int_{z_0}^{z}\exp\left[\pm\left(\frac{s^4}{4}-\frac{\beta s^2}{2}\right)\right]ds, \qquad z_0\leqq z\leqq z_1. \tag{3.14}$$

These integrals satisfy the differential equations:

$$\frac{d^2H_\pm}{dz^2}=\pm(z^3-\beta z)\frac{dH_\pm}{dz}. \tag{3.15}$$

It can be shown that $H_\pm(z,\beta)$ are related to the modified Bessel functions K_n, I_n

(Abramowitz and Stegun (1965)). It can also be shown that, for β large, $H_{\pm}(z) \sim E(\tilde{z}(z))$, where $\tilde{z}(z)$ is a regular function of z.

4. Fixed limit cycles: Stationary asymptotic solutions. In this section, we construct formal asymptotic solutions of the time independent versions of (2.12)–(2.13) for fixed limit cycles. Namely, μ is bounded away from any bifurcation values. In the next section, we allow μ to vary and consider the case in which deterministic bifurcations occur.

4.1. Unstable limit cycle. The unstable limit cycle is analogous to the unstable steady state treated previously (Mangel, 1979). In analogy with that work, a solution of the stationary version of (2.13) is sought in the form:

$$u(x) = \sum \varepsilon^n g^n(x) E(\psi(x)/\sqrt{\varepsilon}) + \varepsilon^{n+1/2} h^n(x) E'(\psi/\sqrt{\varepsilon}). \tag{4.1}$$

In (4.1), $g^n(x)$, $h^n(x)$, and $\psi(x)$ are to be determined. When derivatives are evaluated, (3.9) is used to replace $E''(\psi/\sqrt{\varepsilon})$ by $-E'(\psi/\sqrt{\varepsilon}) \cdot \psi/\sqrt{\varepsilon}$. After substitution into (4.2), terms are collected according to powers of ε. We obtain:

$$\begin{aligned} 0 = \sum_{n=0} & \varepsilon^{n-1/2}(g^n - \psi h^n)\left(b^i\psi_i - \frac{a^{ij}}{2}\psi_i\psi_j\psi\right)E'(\psi/\sqrt{\varepsilon}) \\ & + \varepsilon^n\left(b^i g_i^n + \frac{a^{ij}}{2} g_{ij}^{n-1} + c^i g_i^{n-1}\right)E(\psi/\sqrt{\varepsilon}) \\ & + E'(\psi/\sqrt{\varepsilon})\varepsilon^{n+1/2}\Big\{b^i h_i^n + a^{ij} g_i^n \psi_j + \frac{a^{ij}}{2} g^n \psi_{ij} + g^n c^i \psi_i \\ & \qquad - c^i h^n \psi_i \psi + c^i h_i^{n-1} - \psi a^{ij} h_i^n \psi_j + \frac{a^{ij}}{2} h_{ij}^{n-1} - \frac{a^{ij}}{2} h^n ((\psi\psi_i)_{,j})\Big\}. \end{aligned} \tag{4.2}$$

If a superscript is less than zero, that term is set equal to zero. The leading term, $n = 0$, vanishes if the following equations are satisfied

$$b^i\psi_i - \frac{a^{ij}}{2}\psi_i\psi_j\psi = 0, \tag{4.3}$$

$$b^i g_i^0 = 0, \tag{4.4}$$

$$b^i h_i^0 + \frac{a^{ij}}{2} g^0 \psi_{ij} + a^{ij} g_i^0 \psi_j - h_i^0 a^{ij} \psi_j \psi + g^0 c^i \psi_i - c^i h^0 \psi \psi_i - \frac{a^{ij}}{2} h^0((\psi\psi_i)_j) = 0. \tag{4.5}$$

The transformation $\phi = -\frac{1}{2}\psi^2$ converts (4.3) to the Hamilton–Jacobi or eikonal equation

$$b^i\phi_i + \frac{a^{ij}}{2}\phi_i\phi_j = 0. \tag{4.6}$$

The interpretation of the eikonal equation (4.6) in the stochastic context is discussed by Ventcel and Freidlin (1970), Ludwig (1975), Magnel and Ludwig (1977), and Mangel (1979).

An argument using Hamilton–Jacobi theory (Mangel, 1977, 1979), shows that $\phi = \psi = 0$ on the limit cycle U. Since ϕ is constant on U, $\phi_\theta = 0$ there. We differentiate (4.6) with respect to x^k and obtain:

$$b^i_{,k}\phi_i + b^i\phi_{ik} + \frac{a^{ij}_{,k}}{2}\phi_i\phi_j + \frac{a^{ij}}{2}(\phi_{ik}\phi_j + \phi_i\phi_{kj}) = 0. \tag{4.7}$$

Since $\phi = -\frac{1}{2}\psi^2 = 0$ and $\psi = 0$ on U, $\phi_i = -\psi\psi_i = 0$ on U. Thus, (4.7) becomes

$$b^i\phi_{ik} = 0 = \frac{d}{d\theta}\phi. \tag{4.8}$$

Equation (4.7) is differentiated again and rewritten in the (s, θ) coordinate system on U. One obtains:

$$\frac{d}{d\theta}(\phi_{ss}) + 2b^s_{,s}\phi_{ss} = -a^{ss}(\phi_{ss})^2. \tag{4.9}$$

If we set $W = \phi_{ss}^{-1}$, equation (4.9) becomes a linear equation for W:

$$\frac{dW}{d\theta} - 2b^s_{,s}W = a^{ss}. \tag{4.10}$$

The interpretations of ϕ, W are as follows. To leading order, the first exit probability $u(x)$ will be constant if $\phi(x)$ is constant. If x is a point off the limit cycle $\phi(x) \approx \phi_{ss}(\delta s)^2/2$, where δs is the distance from the limit cycle to the point x. Hence, $\phi(x) \approx (\delta s)^2/2\,W$ and level curves of the first exit probability are obtained at distances proportional to $1/\sqrt{W}$. Thus, W can be viewed as a local variance.

We introduce the integrating factor

$$\Gamma(\theta) = \exp\left[\int_0^\theta 2b^s_{,s}\,d\theta'\right] \tag{4.11}$$

and seek a periodic solution (of period Θ) of (4.9). We obtain:

$$W(\theta) = \left[\frac{\Gamma(\theta)}{1-(1/\Gamma(\Theta))}\right]\left[\int_\theta^{\theta+\Theta}\frac{a^{ss}(0, \theta')\,d\theta'}{\Gamma(\theta')}\right]. \tag{4.12}$$

The period Θ of $W(\theta)$ is the period of the deterministic system.

$W(\theta)$ given in (4.12) is the product of two factors. The first factor involves only the deterministic motion, through $b^s_{,s}$. The second term involves a combination of deterministic and stochastic influences.

Now consider (4.4). Since $b^i = dx^i/dt$, equation (4.5) indicates that g^0 is constant on deterministic trajectories. Following Mangel (1979) we set g^0 to be the same constant on all trajectories. This constant is determined so that the leading part of (4.1) satisfies the boundary conditions. We set $u(x) = 0$ if $x \in S_2$ and $u(x) = 1$ if $x \in S_1$. Suppose that S_1, S_2 are level curves of ψ, with $\psi = \psi_1$ on S_1 and $\psi = \psi_2$ on S_2. In (3.9), we set

$$z_0 = \frac{\psi_2}{\sqrt{\varepsilon}}, \qquad z_1 = \frac{\psi_1}{\sqrt{\varepsilon}} \tag{4.13}$$

and then set

$$g^0 = 1/E(\psi_1/\sqrt{\varepsilon}). \tag{4.14}$$

Then, to leading order $u = 0$ on S_2 and $u = 1$ on S_1. If S_1, S_2 are not level curves of ψ, we proceed as follows. Let ψ_1^m be the maximum value of ψ on S_1 and ψ_2^u be the minimum value of ψ on S_2, then set

$$z_0 = \psi_2^u/\sqrt{\varepsilon}, \qquad z_1 = \psi_2^m/\sqrt{\varepsilon}, \qquad g^0 = 1/E(\psi_1^m/\sqrt{\varepsilon}). \tag{4.15}$$

It can be shown that, if ψ is bounded away from zero on S_1 and S_2 then $u(x)$ is exponentially small on S_2 and $1 - u(x)$ is exponentially small on S_1 (Mangel (1979)).

Next, consider equation (4.5). On the unstable limit cycle U, where $\psi=0$, we obtain

$$\frac{dh^0}{d\theta}-\frac{h^0}{2}a^{ij}\psi_i\psi_j=-\left[\frac{a^{ij}}{2}\psi_{ij}+c^i\psi_i\right]g^0. \tag{4.16}$$

The periodic solution of (4.16) is

$$h^0(\theta)=\frac{\int_\theta^{\Theta+\theta} g^0(a^{ij}\psi_{ij}/2+c^i\psi_i)\exp\{-\int^{\theta'}(a^{ij}/2)\psi_i\psi_j ds\}\,d\theta'}{\exp\{-\int_\theta^{\Theta+\theta}(a^{ij}/2)\psi_i\psi_j\,ds\}\{\exp\{-\int_\theta^{\theta+\Theta}(a^{ij}/2)\psi_i\psi_j\,ds\}-1\}}. \tag{4.17}$$

Once h^0 is known on U, it can be determined off U by the method of characteristics (Mangel (1979)).

The leading part of the expansion (4.1) is given by

$$u(x)=g^0E(\psi/\sqrt{\varepsilon})+O(\sqrt{\varepsilon}). \tag{4.18}$$

Hence, once g^0 and ψ are known, it is possible to construct contours of equal probability of first exit.

4.2. Stable limit cycle. Now consider the case of a stable limit cycle, so that we seek a solution of the stationary version of (2.12):

$$\frac{\varepsilon}{2}(a^{ij}v)_{ij}-((b^i+\varepsilon c^i)v)_i=0. \tag{4.19}$$

Our treatment is slightly different from that of Ludwig (1975). We seek a Gaussian solution of the form

$$v(x)=e^{-\psi(x)2/\varepsilon}(z_0+\varepsilon z_1+\cdots). \tag{4.20}$$

After evaluation of derivatives and substitution into (4.19), terms are collected according to powers of ε (see Ludwig (1975)). The leading term will vanish if ψ satisfies

$$b^i\psi_i+\frac{a^{ij}}{2}\psi_i\psi_j\psi=0. \tag{4.23}$$

Note the change in sign in going from (4.3) to (4.23). If $\phi=\frac{1}{2}\psi^2$, we obtain the eikonal equation (4.6), so that the analysis of (4.23) is identical to the analysis in the previous section. We find

$$v(x)\sim z_0\exp\left[\frac{-\phi_{ss}(\delta s)^2}{2\varepsilon}\right]=z_0\exp\left[\frac{-(\delta s)^2}{2\varepsilon W}\right]. \tag{4.24}$$

Again, $\varepsilon W/2$ has the interpretation of a local variance about the stable limit cycle. Such an interpretation has been given by Ludwig (1975).

The function $z_0(x,t)$ can be determined by solving an ordinary differential equation along the characteristics of (4.23) (see Ludwig, 1975, where details are given). Once z_0 and ϕ_{ss} are known, the leading contribution to $v(x)$ is given by equation (4.24).

5. Hopf bifurcation. The analysis of the preceding section is essentially a linear one. It breaks down at the Hopf bifurcation point, because the linear dynamics vanish at the bifurcation point. The result of § 3 indicates a possible form of the correct asymptotic solution. The Hopf problem to be considered here is analogous to the point source problem for the wave equation (Hadamard (1952)). One solution of that problem, using Hadamard's technique, was given by Zauderer (1971). Although Hadamard's method could also be used for this problem, the construction given here is somewhat simpler.

5.1. Unstable limit cycle, stable focus. We seek an asymptotic solution of the stationary backward equation of the form

$$u(x)=\sum \varepsilon^n g^n(x)H(\psi/\varepsilon^{1/4},\beta/\varepsilon^{1/2})+\varepsilon^{n+3/4}h^n(x)H'(\psi/\varepsilon^{1/4},\beta/\varepsilon^{1/2}), \tag{5.1}$$

where $H(z,\beta)=H_{-}(z,\beta)$, defined in (3.14). When derivatives are evaluated, (3.15) is used to replace H'' by $-H'(\psi^3-\beta\psi)/\varepsilon^{3/4}$. We assume that β has an asymptotic expansion

$$\beta=\sum \varepsilon^k\beta_k. \tag{5.2}$$

After terms are collected according to powers of ε, we obtain:

$$\begin{aligned}
0=&\sum_{n=0}\varepsilon^{n-1/4}H'(\psi/\varepsilon^{1/4},\beta/\varepsilon^{1/2})\left[b^i\psi_i-\frac{a^{ij}}{2}\psi_i\psi_j(\psi^3-\beta_0\psi)\right][g^n-h^n(\psi^3-\beta_0\psi)]\\
&+\sum_{n=0}\varepsilon^nH(\psi/\varepsilon^{1/4},\beta/\varepsilon^{1/2})\left[b^ig_i^n+\frac{a^{ij}}{2}g_{ij}^{n-1}+c^ig_i^{n-1}\right]\\
&+\sum_{n=0}\varepsilon^{n+3/4}H'(\psi\varepsilon^{1/4},\beta/\varepsilon^{1/2})\Big\{b^ih_i^n+c^ig^n\psi_i-c^i\psi_ih^h(\psi^3-\beta_0\psi)\\
&\qquad+\frac{a^{ij}}{2}(2g_i^n\psi_j+g^n\psi_{ij}+h_{ij}^{n-1}-2h_j^n\psi_i(\psi^3-\beta_0\psi)\\
&\qquad-h_i^n\psi_j(\psi^3-\beta_0\psi)-h^n(\psi_i(\psi^3-\beta_0\psi))_j\\
&\qquad+-\sum_{k=1}^{n+1}(\psi\beta_k)\\
&\qquad\qquad\times\left\{\frac{a^{ij}}{2}\psi_i\psi_j\left(g^{n+1-k}-h^{n+1-k}(\psi^3-\beta_0\psi)\right.\right.\\
&\qquad\left.\left.+h^{n+1-k}\left(b^i\psi_i-\frac{a^{ij}}{2}\psi_i\psi_j(\psi^3-\beta_0\psi)\right)\right)\right\}\\
&\qquad+\sum_{k=2}^{n+1}\frac{a^{ij}}{2}\psi_i\psi_jh^{n-k+1}\left(\sum_{j=1}^{k-1}\psi\beta_j(\psi\beta_{k-j})\right)\\
&\qquad-\sum_{k=1}^{n}\psi\beta_k\left(c^i\psi_ih^{n-k}+\frac{a^{ij}}{2}(2h_i^{n-k}\psi_j+h^{h-k}\psi_{ij})\right)\\
&\qquad\qquad-\sum_{k=1}^{n}\beta_k\frac{a^{ij}}{2}\psi_i\psi_jh^{n-k}\Big\}.
\end{aligned}\tag{5.3}$$

In (5.3), if a superscript is less than zero, that term is set equal to zero. The leading term, $n=0$, is composed of three parts and vanishes if

$$b^i\psi_i-\frac{a^{ij}}{2}\psi_i\psi_j(\psi^3-\beta_0\psi)=0 \tag{5.4}$$

$$b^ig_i^0=0 \tag{5.5}$$

$$\begin{aligned}
&b^ih_i^0+\frac{a^{ij}}{2}g^0\psi_{ij}-(\psi^3-\beta_0\psi)a^{ij}h_i^0\psi_j-\frac{a^{ij}}{2}h^0\psi_{ij}(\psi^3-\beta_0\psi)\\
&-h^0\frac{a^{ij}}{2}\psi_i\psi_j(3\psi^2-\beta_0)-(\psi\beta_1)f^0(\psi,x)+g^0c^i\psi_i-c^i\psi_ih^0(\psi^3-\beta_0\psi)=0.
\end{aligned}\tag{5.6}$$

In (5.6), we have introduced the notation that

$$f^n(\psi, x) \equiv \sum_{k=1}^{n+1} \frac{a^{ij}}{2}\psi_i\psi_j(g^{n+1-k} - h^{n+1-k}(\psi^3 - \beta_0\psi)) \tag{5.7}$$
$$+ h^{n+1-k}\left(b^i\psi_i - \frac{a^{ij}}{2}\psi_i\psi_j(\psi^3 - \beta_0\psi)\right).$$

First consider (5.4). Since b^i vanishes at the stable focus P, we set $\psi^3 - \beta_0\psi = 0$ at the focus. This will insure that ψ will have nonvanishing first derivatives. On the limit cycle U, we also set $\psi^3 - \beta_0\psi = 0$. Since $u(U) > u(P)$ we require that

$$\psi(P) = 0, \qquad \psi(U) = \sqrt{\beta_0}. \tag{5.8}$$

When the limit cycle and focus coalesce, we obtain $0 = \sqrt{\beta_0}$, i.e., $\beta_0 = 0$. The singularities of $F(\psi) = \psi^3 - \beta_0\psi$ now match the singularities of the deterministic system. Because the singularities are matched, the present method can be used to produce a uniform solution.

The value of β_0 is still undetermined. It can be calculated by the following iterative procedure. Since (5.4) is a first order partial differential equation, the method of characteristics can be used to solve it, starting just off U, where $\psi = \sqrt{\beta_0^{(0)}}$ and $\beta_0^{(0)}$ is the initial estimate for β_0. We follow characteristics that approach P. If ψ does not approach 0, then $\beta_0^{(0)}$ must be replaced by a better estimate $\beta_0^{(1)}$. The method of false position can be used to calculate iterates of β_0. In this fashion, β_0 can be determined to any order of accuracy. An alternative procedure would follow characteristics from P to U. The choice of method must be made on the basis of numerical practicality.

Although (5.4) can be solved by the method of characteristics, our main interest is in experiments beginning near U. Consequently, we determine ψ in a vicinity of the limit cycle by a Taylor expansion. We assume that $\beta_0 > 0$. Equation (5.4) is differentiated with respect to x^k and then changed to (s, θ) coordinates. We obtain an equation of the form:

$$\frac{d\psi_s}{d\theta} + b^s_{,s}(0, \theta, \mu)\psi_s - a\beta_0\psi_s^3 = 0. \tag{5.9}$$

In deriving (5.9), we have used the fact that $\psi = \sqrt{\beta_0}$ on U (so that $3\psi^2 - \beta_0 = 2\beta_0$ on U). The coefficient a in (5.9) is obtained by a suitable transformation of the matrix a^{ij}. Equation (5.9) is a version of Abel's equation (Davis (1962)). We introduce a new variable z, defined by

$$\psi = 1/Bz \quad \text{where } dB/d\theta = b^s_{,s}B. \tag{5.10}$$

The periodic solution of (5.9) is then

$$\psi_s(0, \theta, \mu) = \left\{-2\beta\int_\theta^{\theta+\Theta} \frac{a(s)}{B(s)^2}\,ds - \frac{\beta\int_0^\Theta a(s)/B(s)\,ds}{\Gamma(\Theta) - 1}\right\}^{-1} \tag{5.11}$$

where

$$B(s) = \exp\left[\int_s^{\Theta+s} b^s_{,s}(0, \theta, \mu)\,d\theta\right], \tag{5.12}$$

$$\Gamma(s) = 1/B(s). \tag{5.13}$$

Equation (5.5) indicates the g^0 is a constant. The value of g^0 can be determined exactly as in § 4. Equation (5.6) is analogous to (4.5). It is slightly more complicated since it

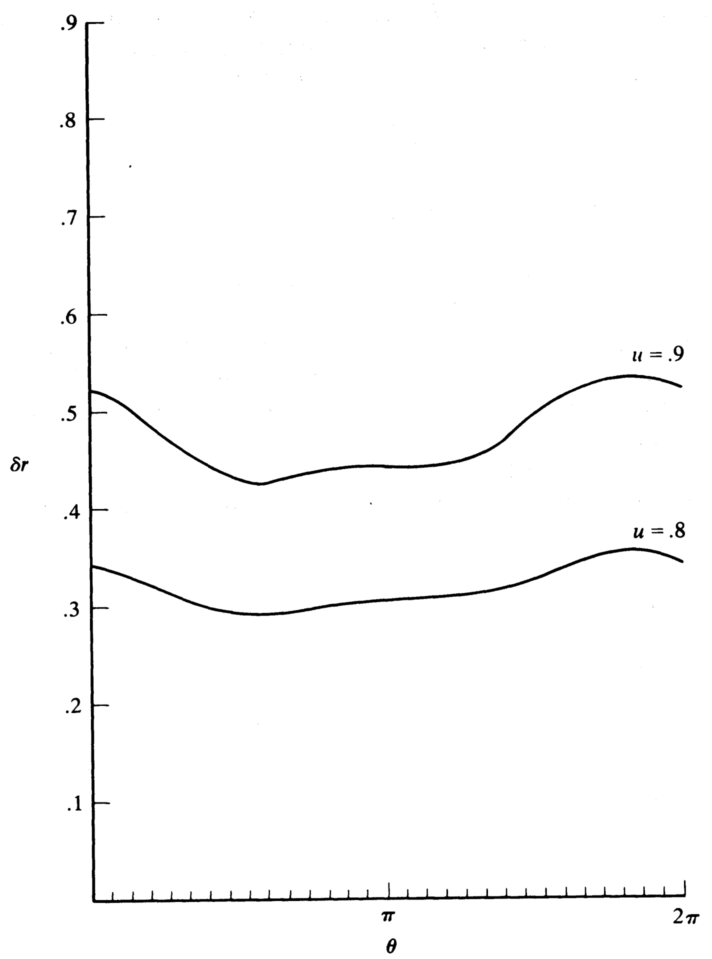

FIG. 3. *Equal probability contours for Example* 6.1.

contains the unknown parameter β_1. This parameter can be determined in the same fashion as β_0 was determined.

It can be shown that all of these constructions are regular at the bifurcation point $\mu = 0$. The proof is analogous to the proof given in Mangel (1977), for bifurcations involving multiple steady states.

TABLE 1
Comparison of theoretical and Monte Carlo results for Example 6.1.

Initial point	u (Theory)	u (Monte Carlo)*
(1.10, 0)	.60	.57
(1.21, .21)	.70	.68
(1.50, .42)	.90	.92
(1.30, 3.97)	.80	.79
(1.53, 5.86)	.90	.92
(1.08, 1.88)	.60	.58
(1.19, 2.30)	.70	.69

* 2,500 simulations were performed.

5.2. Stable limit cycle, unstable focus. In this case, we are interested in uniform solutions of the forward equation (4.19). It is clear that the Gaussian ansatz in § 4.2 breaks down for μ small. We seek a solution of the form

$$v(x) \sim \exp\left[-\frac{1}{\varepsilon}\left(\frac{\psi(x)^4}{4} - \beta\frac{\psi(x)^2}{2}\right)\right](z^0(x) + \varepsilon z^1(x) + \cdots). \tag{5.14}$$

Following the procedure of § 4.2, we are led to

$$b^i\psi_i + \frac{a^{ij}}{2}\psi_i\psi_j(\psi^3 - \beta_0\psi) = 0. \tag{5.15}$$

Equation (5.15) can be treated by the method of characteristics or by a Taylor expansion. The function $z^0(x)$ can be determined by integration along the characteristics of (5.15) (Ludwig (1975)). The determination of the rest of $v(x)$ proceeds as in the previous section.

Thus, the stationary distributions for the Hopf bifurcation problem have been determined. These distributions are regular functions of μ, the deterministic bifurcation parameter.

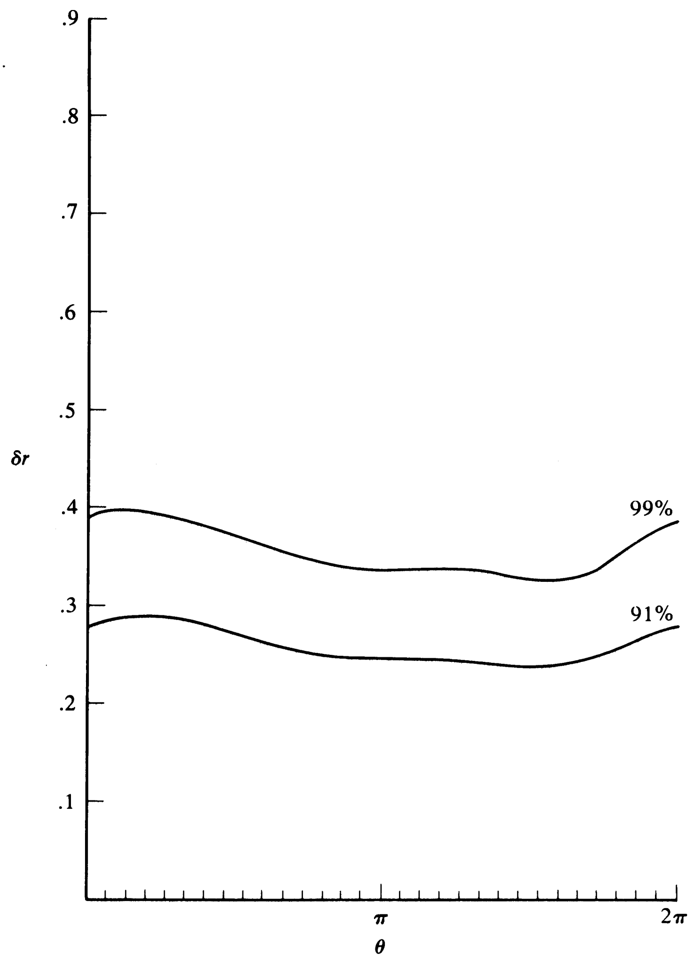

FIG. 4. *Equal probability confidence contours for Example 6.2.*

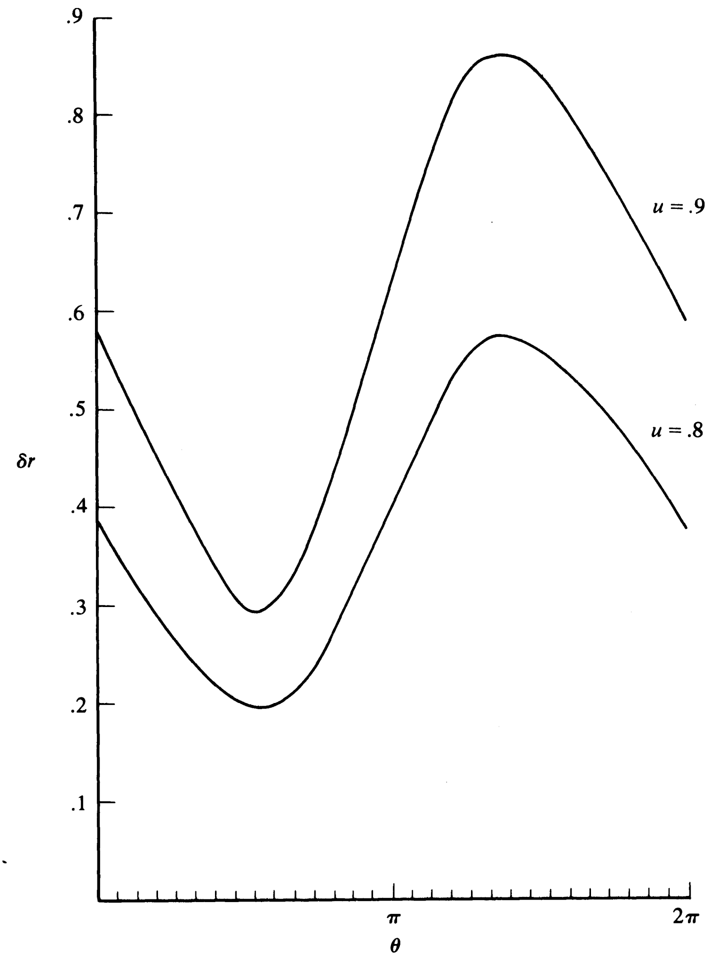

FIG. 5. *Equal probability contours for Example* 6.3.

6. Numerical results for fixed limit cycles. In this section, we present a number of numerical examples that illustrate the behavior of $u(x)$ and $v(x)$, as determined in § 4. For convenience, we use systems already in polar coordinates.

The theoretical results reported below for $u(x)$ were calculated using the first term ($n = 1$) in the expansion (4.1). The theoretical results reported below for $v(x)$ were calculated using the term $e^{-\psi^2/\varepsilon} z_0$ from (4.20). The Monte Carlo results reported below were obtained by numerical solution of the Ito equation. Use of the Ito equation is equivalent to assuming that $Y(s)$ in (2.10) is white noise.

Example 6.1. In this case, the following deterministic dynamics were assumed:

$$\dot{r} = r(r-1)(2-r)(1.1+\cos\theta), \tag{6.1}$$

$$\dot{\theta} = 1 \tag{6.2}$$

with covariance matrix elements zero, except for a^{rr}:

$$\varepsilon a^{rr} = .1[r^2+(2-r)^2](1.5+\cos\theta)^2 \tag{6.3}$$

TABLE 2
Comparison of theoretical and Monte Carlo results for Example 6.3.

Initial point	u (Theory)	u (Monte Carlo)*
(1.14, 0)	.60	.62
(1.22, .21)	.70	.73
(1.49, .42)	.90	.90
(1.55, 3.97)	.80	.82
(1.66, 5.86)	.90	.87
(1.12, 1.88)	.70	.67
(1.06, 1.47)	.60	.64

* 2,500 simulations were performed.

The circle $r=1$ is an unstable limit cycle. Let

$$u(x)=Pr\{\text{process hits } r=1.98 \text{ before } r=.02|\tilde{x}(0)=x\}. \tag{6.4}$$

In Fig. 3, we show the $u=.8, .9$ contours $\delta r(\theta)$ where θ measures distance along the cycle and δr is the distance from $r=1$ to the contour. The noise and deterministic

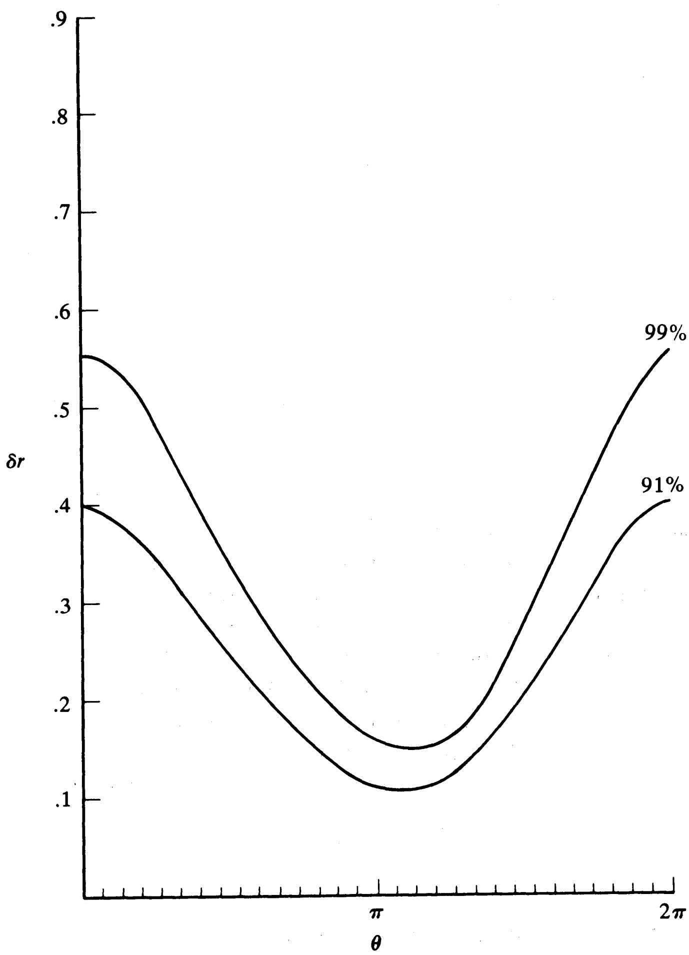

FIG. 6. *Probability confidence contours for Example* 6.4.

dynamics are in phase. In Table 1, we compare the theory with Monte Carlo experiments.

Example 6.2. We now consider the system with deterministic dynamics

$$\dot{r} = r(r-1)(r-2)(1.1+\cos\theta), \tag{6.5}$$

$$\dot{\theta} = 1 \tag{6.6}$$

with εa^{rr} given by (6.3). The deterministic system has a stable limit cycle at $r = 1$. Let

$$v(x)\,dx = Pr\{\text{process is eventually found between } (x, x+dx)\}. \tag{6.7}$$

In Fig. 4, we plot the .91, .99 confidence contours of $v(x)$ as a function of $\delta r(\theta)$, where δr is the distance from the limit cycle to the contour and θ measures distance along the limit cycle.

Example 6.3. We now take $\dot{\theta}$ as above and the following deterministic dynamics for r:

$$\dot{r} = r(r-1)(2-r)(1.1+\sin\theta) \tag{6.8}$$

with εa^{rr} given by (6.3). In this case, the noise is out of phase with the deterministic cycling. In Fig. 5, we plot the $u = .8, .9$ contours and in Table 2, compare Monte Carlo and theoretical results.

Example 6.4. We take for the deterministic dynamics

$$\dot{r} = r(r-1)(r-2)(1.1+\sin\theta), \tag{6.9}$$

$$\dot{\theta} = 1 \tag{6.10}$$

with εa^{rr} given by (6.3).

In this case, $r = 1$ is a stable limit cycle. In Fig. 6, the .91 and .99 confidence contours are plotted.

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