Search and Stock Depletion: Theory and Applications

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A general theory for the estimation of stock size from search data is developed. In the theory, it is assumed that discrete aggregations (schools of fish, beds of clams) are encountered. The search model is an extension of the Poisson process to include depletion. The theory provides a way of estimating stock size and confidence intervals around the estimate, as well as the ability to predict future catches for a given level of effort. Three applications of theory are described: (1) estimating stock size when there is no catch; (2) determining, in real time, the length of fishing seasons; and (3) an empirical study of stock assessment of Pacific ocean perch (*Sebastes alutus*) near Rennell Sound, British Columbia.

Nous avons mis au point une théorie générale sur l'estimation des stocks d'après des données de recherche. Cette théorie suppose qu'on rencontre des groupes séparés (bancs de poissons, colonies de clams). Nous avons élargi le procédé de Poisson pour tenir compte de l'amenuisement. La théorie permet d'estimer l'importance des stocks, l'intervalle de confiance de l'estimation et la possibilité de prévoir les prises en fonction d'un effort de pêche donné. Nous décrivons trois applications : (1) estimation des stocks lorsqu'il n'y a pas de prises ; (2) détermination, en temps réel, de la longueur de la saison de pêche ; et (3) étude empirique de l'évaluation des stocks de sébastes (*Sebastes alutus*) près de la baie Rennell en Colombie-Britannique.

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1. Introduction

One of the thorniest problems in the management of fisheries is the problem of determining stock abundance. For this reason, biomass surveys are often run by the managing organizations. In some fisheries, the cost of the survey is about the same as the economic value of the fishery (C. Clark, Mathematics Department, University of British Columbia, Vancouver, B.C., pers. comm.).

In those cases without a stock survey, some other procedure for estimating stock size is needed. Perhaps the most common is the model assuming a linear relationship between catch and population:

$$H = qEP$$

where H is harvest, E is fishing effort, P is population size, and q is a proportionality coefficient (the catchability coefficient). With this model H/E, catch per unit effort (CPUE) is proportional to population. To use this model, one needs some measure of fishing effort, a quantity that is notoriously hard to measure in any real fishery (e.g. see Rothschild 1972; Wilen 1979). Even if effort could be measured accurately, the CPUE model does not always produce measures of population size that reflect the population (e.g. see Clark and Mangel 1979).

One case in which CPUE is expected to break down is when the fishermen spend considerable amounts of time searching for dense schools or aggregates of fish (this is actually the case considered by Clark and Mangel 1979 for tuna). In this case, the search effort is as important to the estimate of population size as is the time spent harvesting, so that one should consider the relationship between search effort and stock size (Mangel 1982). There is a growing body of literature on the role of search in fisheries (e.g. see Allen and Punsly 1984; Butterworth 1982; Cooke 1984; Neyman 1949; Paloheimo 1971; Rothschild 1977). Allen and Punsly (1984), in particular, advocated the concept of using catch per hour searching as a way of estimating abundance. They used a very simple search model, followed by regression techniques.

In this paper, we introduce a new theory for the estimation of stock size using search data. The model and development of the theory are presented in section 2. Sections 3, 4, and 5 are management oriented. Section 3 considers the problem of estimating the stock size when there is no catch. In section 4, we show how search theory can be used to provide real time determination of the length of fishing seasons. Section 5 contains the results of an empirical study on Pacific ocean perch (*Sebastes alutus*) off Rennell Sound, British Columbia, for the estimation of stock size and prediction of harvest as a function of effort. Throughout the paper, we are concerned with search and stock depletion in a single fishing season, so we ignore recruitment. The theory that we develop, however, can be extended to include recruitment. Section 6 contains comments on implementation of this work, as well as a discussion of some of the extensions currently under investigation.

The innovative aspects of the work presented in this paper include the following:

(1) A general theory that relates stock abundance to search rates and includes depletion of the stock due to harvesting (cf. Allen and Punsly 1984 where depletion is ignored). This theory involves two parameters: stock size, N, and a measure of search effectiveness, ϵ.

(2) An operational definition of ϵ that is independent of any data set that one would analyze. The parameter ϵ is related to operational variables such as vessel speed, detection width, and size of the region being searched.

(3) A combination of Bayesian and maximum likelihood methods for estimating the parameter N. In particular, maximum likelihood methods are used to provide an estimate of N, $\hat{N}$, and Bayesian confidence intervals are then derived. The Bayesian methods are particularly useful when there is a lack of data; e.g. how does one estimate stock abundance in the absence of catch.

2. Search Model and Theory

Perhaps the simplest search model is the Poisson process. In this model, one assumes that the search process is random and that

$$\begin{aligned}(2.1)\quad &\text{Prob \{find another school in the next } \Delta t\} = \lambda\Delta t \\ &\text{Prob \{do not find another school in the next } \Delta t\} = 1 - \lambda\Delta t \\ &\text{Prob \{anything else\}} = 0\end{aligned}$$

where Δt is a small time interval and λ is a parameter. If the search is random and only one school is present, λ can be estimated using classical search theory (Koopman 1980), which gives

$$(2.2)\quad \lambda = \frac{Wv}{A}$$

where W is the sweep width of the searcher (for purposes of fisheries, W can be thought of as the area of the net opening or, the detection width of the echosounder), v is the speed of the searcher, and A is the area (or volume) of water where the school might be.

It is worthwhile to examine in detail the assumptions that lead to (2.1). The most important of these are (i) cohesive and identifiable schools, (ii) random search, and (iii) no depletion of the stock. Certainly, not all types of fish move in cohesive and identifiable schools but there are a sufficient number of species that do (tuna, some groundfish, whales) to make a theory concerned with the search for schools sufficiently interesting. It is not crucial that the fish spend all of the time in schools, only that the search is for schools. It is also true that fishermen probably do not execute random searches. This, however, does not mean that the ultimate search process cannot be modeled as random. First, the schools being sought may move randomly or quasi-randomly. Second, on the small scale of tracklines, ostensibly nonrandom searches take on considerable fluctuations (Koopman 1980). Third, there are experiments to indicate the generality of exponential models for times between detections (equivalent to a random search). These experiments (Washburn 1981, p. 2–8) involved a computer joystick game in which an intelligent pursuer sought an intelligent evader. The experiments indicate that a random search model provides an excellent description of the overall detection process. Fourth, if there is considerable set time after a discovery, the information gained in locating a school may be dissipated so that the search for the next school is random. Allen and Punsly (1984) examined the hypothesis of random search and concluded that it is acceptable. They also pointed out that data often exhibit "overdispersion," i.e. too may intervals between detections that are too long or too short. Mangel and Clark (1983) showed that such deviations from pure negative exponential distributions can arise in the following way. Assume that locally, search is random with parameter λ, but that λ has a distribution. For example, if one examines $1^0 \times 1^0$ cells, then in each cell it could be assumed that, conditioned on λ, the number of schools found in $(0, t)$ is Poisson with parameter λt. If λ has a distribution, say a gamma distribution, then the unconditional distribution of catch is negative binomial, which exhibits overdispersion, relative to the Poisson distribution. The underlying search mechanism, however, is still random. The assumption of random search will be used in this paper. The third assumption leading to (2.1) is the lack of depletion. This makes no sense if one is trying to estimate the effects of harvesting on stock abundance. In particular, if depletion is not taken into account, there is a tendency to considerably overestimate abundance.

In this paper, a methodology is introduced under the following assumptions:

(i) The search process involves the search for discrete and identifiable schools. (This does not imply, however, that the fish being sought always school — only during the search process. An example is given in section 5.)

(ii) The times between detections are independent, exponential random variables (thus corresponding to random search); the distribution of these variables changes due to search.

(iii) When a detection occurs, the entire school is fished out. The parameter characterizing the search rate is then decremented.

Further discussion, and ways to modify these assumptions, is found in section 6. Although it may appear that these assumptions are too restrictive to provide any useful results, this is not the case as the work (including an empirical study) reported in sections 3–5 demonstrates. Clearly, one would like a more general theory, but there must always be a starting point.

To modify (2.1) to include depletion, we keep the assumption of random search but now assume that many schools (or aggregations) are present and that detections of different schools are independent events. With these assumptions, a reasonable modification of (2.1) is as follows:

$$\begin{aligned}(2.3)\quad &\text{Prob \{find another school in the next } \Delta t | i \text{ were found thus far\}} = (\lambda - i\epsilon)\Delta t \\ &\text{Prob \{do not find another school in the next } \Delta t | i \text{ were found thus for\}} = 1 - (\lambda - i\epsilon)\Delta t.\end{aligned}$$

There are two parameters in this model: λ and ϵ. Since λ is proportional to the initial rate of detections, it is really unknown. The situation is better with ϵ, however. Suppose that there were exactly n schools present initially. Then the independence assumption leads to

$$\lambda = n\frac{Wv}{A}.$$

After the first detection, there are $n - 1$ schools present, so λ

would be replaced by $(n - 1)Wv/A$. Consequently

$$(2.4) \qquad \epsilon = \frac{Wv}{A}.$$

The first contribution of search theory is to provide this expression for ϵ in terms of operational variables (cf. Goudie and Goldie 1981). Note that ϵ is a measure of search ability. An example of the calculation of the sweep width W is given in section 5. Observe too that ϵ, as defined by (2.4), can be computed outside of the data being analyzed.

When k searchers are present and search independently, (2.3) is modified by replacing $(\lambda - i\epsilon)$ by $k(\lambda - i\epsilon)$. With these assumptions, one finds that the number of schools discovered has a binomial distribution with parameters $N = \lambda/\epsilon$ and $p = 1 - e^{-k\epsilon t}$. Thus

$$(2.5) \qquad \begin{aligned} &\Pr\{k \text{ searchers discover } n \text{ schools in } (0, t)\} \\ &\equiv p(k, n, t, \epsilon) \\ &= \binom{\lambda/\epsilon}{n}(1 - e^{-k\epsilon t})^n(e^{-k\epsilon t})^{(\lambda/\epsilon - n)}, \ n = 0, 1, \ldots, \lambda/\epsilon \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Equation (2.5) is derived by standard probabilistic methods (e.g. Feller 1968). According to (2.5), the expected number of schools discovered, $E_t(n)$, and variance in the number of schools discovered, $V_t(n)$, in the interval $(0, t)$ are

$$(2.6) \qquad \begin{aligned} E_t(n) &= \frac{\lambda}{\epsilon}(1 - e^{-k\epsilon t}) \\ V_t(n) &= \frac{\lambda}{\epsilon}(1 - e^{-k\epsilon t})e^{-k\epsilon t}. \end{aligned}$$

Note that λ/ϵ can be interpreted as the initial number of schools. Equations (2.5) and (2.6) refer to the search time only. If each aggregation requires a harvest time τ_h, the total time spent fishing and searching is $T = t + n\tau_h$. Since n is a random variable, then so is T, and its distribution function is easily found from (2.5). In most cases, however, the inverse problem is of more interest, i.e. given T find the estimates of t and n. This problem is discussed in section 5.

Equations similar to (2.5) and (2.6) were derived by DeLury ((1947, 1951); Seber (1982, p. 296)) in his classic work on stock estimation. The difference is that in the current model, ϵ is given in terms of operational parameters (area searched, vessel speed, and sweep width) and can thus be determined independently of the data being analyzed. Without the operational interpretation of ϵ, the estimation problems become much more difficult. For example, the methods of Leslie and DeLury (see Seber 1982, p. 296, for a good discussion) involve regressions that seek to determine both the initial size of the population and the catchability coefficient, which is analogous to ϵ. These methods can lead to negative estimates of stock abundance, something that never happens with the methods advocated in this paper. Thus, there is considerable gain made by using an operational definition of ϵ that is independent of the actual fishing process.

In general, λ (or λ/ϵ) will be unknown. Two different methods for estimating λ, when ϵ is known, are given below. One method fixes the number of schools detected in a variable time. The other method fixes the time interval and allows the number of schools detected to vary. For simplicity, assume that there is only one searcher, so that $k = 1$. Suppose that this searcher discovers n schools and that T_i, $i = 1, \ldots, n$ was the observed time between the $(i - 1)$st and ith detections. The total search time T is

$$\sum_{i=1}^{n} T_i.$$

According to the search model (2.3), the random variable $\tilde{T}_i$, which is the time between the $(i - 1)$st and ith detections, is exponentially distributed with parameter $\lambda - (i - 1)\epsilon$. The $\tilde{T}_i$, $i = 1, \ldots, n$ are independent but not identically distributed random variables. The likelihood, $\mathscr{L}$, of the set $\{T_1, \ldots, T_n\}$ is then

$$(2.7) \qquad \mathscr{L} = \prod_{i=1}^{n}(\lambda - (i - 1)\epsilon)e^{-(\lambda - (i-1)\epsilon)T_i}.$$

The maximum likelihood estimator (MLE) for λ is derived from (2.7). Taking the derivative of the logarithm of $\mathscr{L}$ gives

$$(2.8) \qquad \frac{\partial}{\partial\lambda}\log\mathscr{L} = \sum_{i=1}^{n}\left\{\frac{1}{\lambda - (i - 1)\epsilon} - T_i\right\}.$$

Setting the right hand side of (2.8) equal to zero gives the nth-order algebraic equation

$$(2.9) \qquad \sum_{i=1}^{n}\frac{1}{(\lambda - (i - 1)\epsilon)} = \sum_{i=1}^{n} T_i = T.$$

(Note that setting $\epsilon = 0$ gives $\lambda = n/T$, the MLE for a Poisson process.)

The second derivative of $\log\mathscr{L}$ is given by

$$(2.10) \qquad \frac{\partial^2}{\partial\lambda^2}\log\mathscr{L} = -\sum_{i=1}^{n-1}\frac{1}{(\lambda - (i - 1)\epsilon)^2}.$$

Since the second derivative is negative, the solution of (2.9) is indeed a MLE.

When ϵ is small, the solution of equation (2.9) can be found approximately as a power series in ϵ. This approach gives

$$(2.11) \qquad \lambda = \frac{n}{T} + \left(\frac{n - 1}{2}\right)\epsilon - \frac{(n^2 - 1)T}{12n}\epsilon^2 + 0(\epsilon^3).$$

To employ the second estimate for λ, fix t and set $N = \lambda/\epsilon$. In light of equation (2.5), N is the initial number of schools. Equation (2.5) is then a likelihood equation for N, given that n schools were discovered. In what follows, we set $k = 1$. Results for $k > 1$ are obtained by replacing t by kt. Thus, the likelihood is

$$(2.12) \qquad \begin{aligned} L(N; n) &= \binom{N}{n}(1 - e^{-\epsilon t})^n e^{-\epsilon t(N-n)}, \quad N = n, n + 1, \ldots \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

The problem is now that of estimating N in the binomial distribution on the basis of a single observation on n. Somewhat tongue-in-cheek, this problem is aptly described as "capture – recapture – without recapture." (Other problems in which the binomial parameter N must be estimated are discussed by Blumenthal and Dahiya (1981) and Olkin et al. (1981).)

First consider the MLE for N. It is most easily found through the likelihood ratio

$$(2.13) \qquad \frac{L(N + 1; n)}{L(N; n)} = \frac{\binom{N+1}{n}}{\binom{N}{n}}e^{-\epsilon t} = \left\{\frac{N + 1}{N + 1 - n}\right\}e^{-\epsilon t}.$$

The ratio in (2.13) is 1 if

$$(2.14)\quad \left(\frac{N+1}{N+1-n}\right) e^{-\epsilon t} = 1.$$

Solving for N gives

$$(2.15)\quad N = \frac{n}{1-e^{-\epsilon t}} - 1.$$

From (2.15) it follows immediately that the MLE for N is

$$(2.16)\quad \hat{N} = \left[\frac{n}{1-e^{-\epsilon t}}\right].$$

Here, $[x]$ is the integer part of x. From (2.16), it follows that the estimate for λ is $\lambda_2 = N\epsilon$. The two estimates (based on (2.9) and (2.16)) are identical to within order ϵ (Table 1).

Equation (2.16) is similar to the more common estimate, which reads

$$\text{Biomass} = \frac{\text{Catch}}{1 - \exp\{-q\cdot\text{effort}\}}$$

where q is the catchability coefficient and effort is measured in some aggregate form. The main differences are that in (2.16) ϵ is a completely operational variable and t is search time only. In this sense, (2.16) is much less aggregated than the standard catch equation.

One could, in principle, try to "operationalize" q by defining it in terms of operational parameters (e.g. Rothschild 1972, 1977). There is still the problem of the aggregation and definition of effort, however. The use of equation (2.16) avoids this problem.

Note that $\hat{N}$ depends upon both the catch and the elapsed search time t. From (2.16), one sees that $\hat{N}$ tends to increase with n, if t is fixed. If no aggregates are encountered for a long time, $\hat{N}$ will eventually equal n. This indicates that the entire population was caught.

If a Bayesian view is adopted, the posterior distribution of N (and thus of λ) can be calculated if a prior distribution $g(N)$ and an observation n are given. If $g(N|n)$ denotes the posterior, then

$$(2.17)\quad g(N|n) = \frac{L(N;\, n)g(N)}{\sum_{N=n}^{\infty} L(N;\, n)g(N)}$$

where $L(N;\, n)$ is given by (2.12).

One choice for the prior distribution is the improper prior

$$(2.18)\quad g(N) \equiv \begin{cases} 1 & N = 0, 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

The prior distribution defined in (2.18) is improper because its sum is infinite. The posterior distribution, however, is a true probability distribution as the following calculation shows. For the improper prior (2.18), equation (2.17) becomes

$$(2.19)\quad g(N|n) = \frac{L(N;\, n)}{\sum_{N=n}^{\infty} L(N;\, n)},\quad N = n, n+1, \ldots$$

The identity

$$\sum_{N=n}^{\infty} \binom{N}{n} x^{N-n} = (1-x)^{-n-1}$$

TABLE 1. Comparison of estimates of λ based on fixed n ($\hat{\lambda}_1$) and fixed t ($\hat{\lambda}_2$). Search time = 100 h; k = 1 vessel.

Schools found					
n	ϵ	λ_1	n_{R1}[a]	λ_2	n_{R2}[a]
10	0.01	0.153	5.31	0.150	5
15	0.01	0.232	8.22	0.230	8
20	0.01	0.313	11.1	0.310	11
25	0.01	0.390	14.0	0.390	14
30	0.01	0.470	17.0	0.470	7
10	0.005	0.124	14.9	0.125	15
15	0.005	0.188	22.6	0.190	23
20	0.005	0.252	30.3	0.250	30
25	0.005	0.315	38.1	0.315	38
30	0.005	0.379	45.7	0.379	46

[a] n_{Ri} is the expected number of schools remaining, using estimator λ_i.

allows one to write the denominator in (2.19) as

$$(2.20)\quad \sum_{N=n}^{\infty} L(N;\, n) = \sum_{N=n}^{\infty} \binom{N}{n} (1-e^{-\epsilon t})^n e^{-\epsilon t(N-n)}$$
$$= (1-e^{-\epsilon t})^n (1-e^{-\epsilon t})^{-n-1}$$
$$= (1-e^{-\epsilon t})^{-1}.$$

Consequently, the posterior $g(N|n)$ becomes

$$(2.21)\quad g(N|n) = \binom{N}{n} (1-e^{-\epsilon t})^{n+1} e^{-\epsilon t(N-n)},\quad N = n, n+1, \ldots$$

A second choice is the noninformative prior (Martz and Waller 1982)

$$(2.22)\quad g(N) = \left(\frac{-\partial^2 L(N;\, n)}{\partial N^2}\right)^{1/2} \Bigg|_{\frac{\partial L(N;\, n)}{\partial N} = 0.}$$

A prior is called noninformative if the data only change the location, but not the shape, of the likelihood curve (Box and Tiao 1973, p. 32). The prior in (2.22) can not be computed analytically; but it is easy to find its value numerically. When calculating $g(N)$ in (2.22), the observation n is eliminated by setting $\partial L(N;\, n)/\partial N = 0$ and solving for n in terms of N.

Figures 1A and 1B show the posterior distribution on N using (2.18) and (2.22). Since the results are so similar, the uniform prior will be used. If one sets $p = 1 - e^{-\epsilon t}$ and $q = e^{-\epsilon t}$, then (2.21) becomes

$$(2.23)\quad g(N|n) = \binom{N}{n} p^{n+1} q^{N-n},\quad N = n, n+1, \ldots$$
$$= 0 \qquad \text{otherwise.}$$

This form of (2.21) will be very useful in the sequel. Equation (2.23) can be used to construct Bayesian confidence intervals for N around $\hat{N}$ by solving for the smallest J^* such that

$$(2.24)\quad g(\hat{N}|n) + \sum_{j=1}^{J^*} [g(\hat{N}-j|n) + g(\hat{N}+j|n)] \geq \gamma$$

where γ is the desired confidence level. Equation (2.24) is easily evaluated on a desk-top microcomputer once a prior $g(N)$ is given. Table 2 provides an example of such a confidence interval, for the improper prior (2.18).

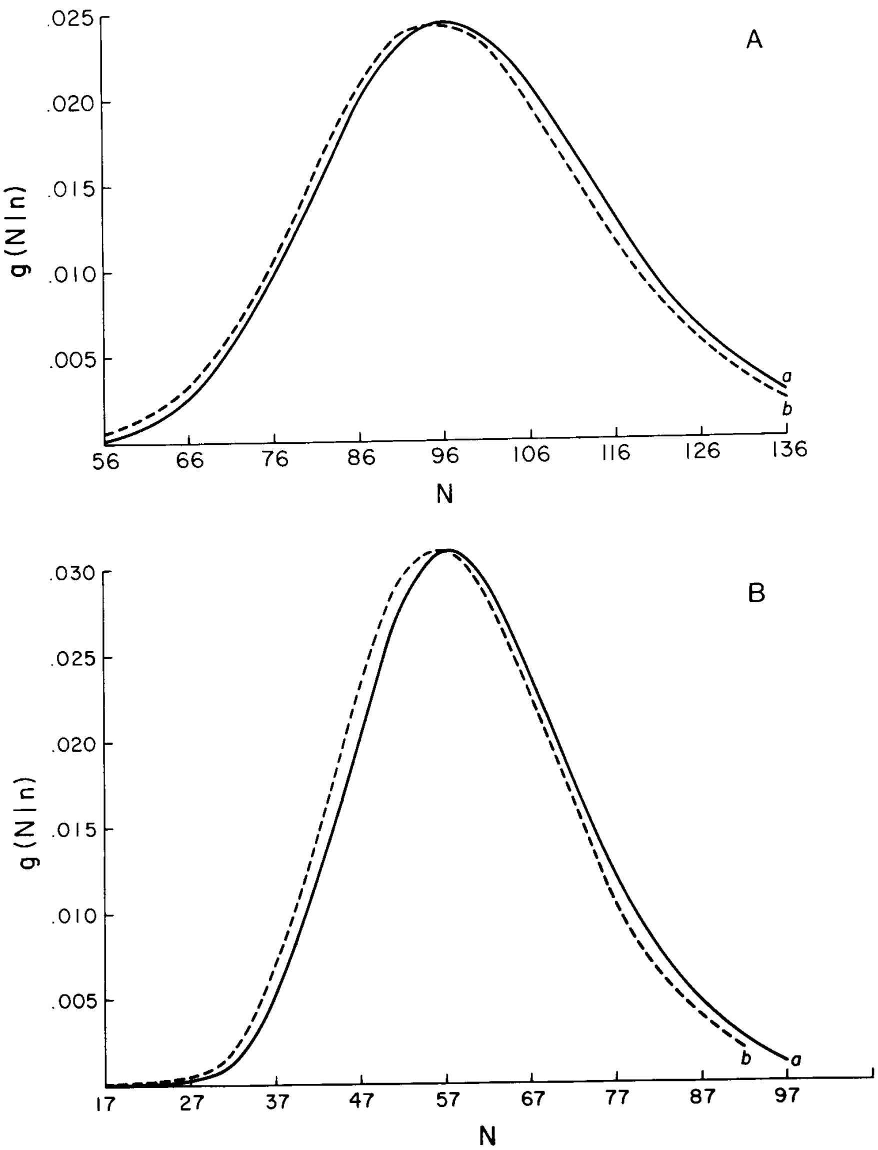

FIG. 1. Comparison of uniform (curve a) and noninformative priors (equation 2.22) (curve b) distributions on N. Parameter values: $\epsilon = 0.01$, $t = 30$, (A) $n = 25$, (B) $n = 15$.

Next consider the variable $Y = N - n$, which is the number of schools that remain. Since $\Pr\{Y = y\} = \Pr\{N = n + y\}$, equation (2.23) shows that

$$(2.25)\quad \Pr\{Y = y\} = \binom{n+y}{n} p^{n+1} q^{y}$$

$$(2.26)\quad = \binom{y+r-1}{y} p^{r} q^{y}$$

where $r = n + 1$. Consequently, the distribution of the remaining number of schools is negative binomial with parameters r and p.

The noninformative prior (2.18) can be converted to a proper prior distribution if we fix a maximum value for N. Let N_m be that value plus one. Then a proper noninformative prior $g(N)$ is given by

$$(2.27)\quad g_m(N) = \begin{cases} 1/(N_m + 1), & N = 0, 1, \ldots, N_m \\ 0 & \text{otherwise.} \end{cases}$$

Given that n schools were observed, it is easily found from (2.17) that

$$(2.28)\quad g_m(N|n) = c \binom{N}{n} q^{N}, \quad N = n, n+1, \ldots, N_m$$

where the normalization constant is

$$1/c = \sum_{N=n}^{N_m} \binom{N}{n} q^{N}.$$

Other choices of the prior distribution $g(N)$ are possible, and equation (2.17) can be tailored to the user's prior intuition about N (an example of such intuition is discussed in section 3, see Fig. 5). Given a prior distribution, equation (2.24) can be used to construct a Bayes confidence interval for N around $\hat{N}$. (If k independent searchers operate, then $1 - e^{-\epsilon t}$ is replaced by $1 - e^{-k\epsilon t}$.)

Finally, consider predictions of future harvest (in the same season) given that n schools were found in $(0, t)$. That is, if s more units of search time elapse, what can be said about the statistics of the predicted catch? Let H denote this catch. In light of (2.5), the conditional expectation and variance of the remaining harvest are

$$(2.29)\quad \begin{aligned} E\{H|\hat{N}, n, s\} &= (\hat{N} - n)(1 - e^{-k\epsilon s}) \\ \mathrm{Var}\{H|\hat{N}, n, s\} &= (\hat{N} - n)(1 - e^{-k\epsilon s})e^{-k\epsilon s} \end{aligned}$$

so that the coefficient of variation of the remaining harvest is

$$(2.30)\quad \mathrm{CV}\{H|\hat{N}, n, s\} = \frac{1}{\sqrt{\hat{N} - n}}\left[\frac{e^{-k\epsilon s}}{1 - e^{-k\epsilon s}}\right]^{1/2}.$$

Consider a fishery manager operating under an exogenous constraint on escapement of the following form: the stock level at the end of the season should be a fraction f $(0 < f < 1)$ of the stock level at the start of the season. For the stochastic problem here, one can only calculate the probability that the stock level at the end of the season is greater than f times the initial stock level. Suppose that n schools have been caught after a search time t. Let $p(f, n, t, s, N)$ be the probability that at least fN schools remain if there are s more time units of search in the season, conditioned on n and N. This probability is the same as the probability that the total harvest is less than $(1 - f)N$, subject to the same conditioning. To find $p(f, n, t, s, N)$, introduce the random variable n_s = number of schools caught in the next s units of search time. The distribution of n_s is binomial with parameters $N - n$ and $1 - e^{-k\epsilon s}$. Thus

$$(2.31)\quad \begin{aligned} p(f, n, t, s, N) &= \Pr\{n + n_s \leq (1 - f)N\} \\ &= \Pr\{n_s \leq (1 - f)N - n\} \\ &= \begin{cases} \displaystyle\sum_{m=0}^{(1-f)N-n} \binom{N-n}{m}(1 - e^{-k\epsilon s})^m(e^{-k\epsilon s})^{N-n-m} & \text{if } n \leq (1-f)N \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The actual harvest statistics must be averaged over the posterior distribution $g(N|n)$; i.e.

$$(2.32)\quad \begin{aligned} E\{H|n, s\} &= \sum_{N=n}^{\infty} g(N|n)E\{H|N, n, s\} \\ \mathrm{CV}\{H|n, s\} &= \sum_{N=n}^{\infty} g(N|n)\,\mathrm{CV}\{H|N, n, s\} \\ p(f, n, t, s) &= \sum_{N=n}^{\infty} g(N|n)p(f, n, t, s, N). \end{aligned}$$

An example showing how (2.32) is used is presented in the next section.

3. Estimating Stock Size in the Absence of Catch

Imagine a region in which some effort, measured by search time, is expended but no fish are caught. What can be said about the stock level in the region? This is a case in which the Bayesian approach is essentially the only one possible. One cannot rule out the presence of fish; instead the quantity that must be found is the posterior probability distribution for N, the number of aggregations in the region, when $n = 0$ schools were discovered during t hours of search.

TABLE 2. An example showing confidence intervals ($n = 20$, $P = 0.05$, $\hat{N} = 400$).

Confidence level (γ)	Range of N
0.85	[339, 459]
0.90	[335, 465]
0.95	[331, 467]
0.99	[327, 471]

Using the noninformative prior (2.18) (which is probably a good assumption if the region has never been fished before) gives the posterior density

$$(3.1)\quad g(N|0) = pq^N \quad N \geq 0$$

where $p = 1 - e^{-k\epsilon t}$ and $q = 1 - p$.

In this case, the MLE for N is $\hat{N} = 0$. Thus, the Bayesian confidence interval (2.24) is of the form $[0, J^*]$, where J^* is the smallest integer such that

$$(3.2)\quad \sum_{N=0}^{J^*} g(N|0) \geq \gamma.$$

Since $g(N|0)$ is simply a geometric distribution, the sum in (3.2) is $1 - q^{J^*+1}$. Thus, (3.2) can be explicitly solved for J^*, giving

$$(3.3)\quad J^* = \left[\frac{\log(1 - \gamma)}{\log q}\right] - 1 = \left[\frac{\log(1 - \gamma)}{-k\epsilon t}\right] - 1$$

where, as before, $[x]$ is the integer part of x.

If the proper noninformative prior (2.27) is used, the posterior distribution (2.28) turns out to be

$$(3.4)\quad \begin{aligned} g_m(N|0) &= \left\{\frac{p}{1 - q^{N_m+1}}\right\} q^N, \quad N = 0, 1, \ldots, N_m \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned}$$

A Bayesian confidence interval for N is now constructed in the same way. It is of the form $[0, J_m^*]$, where J_m^* is given by

$$(3.5)\quad J_m^* = \left[\frac{\log((1 - \gamma)(1 - q^{N_m+1}))}{\log q}\right] - 1.$$

Figures 2A and 2B show results of calculations using $p = 1 - e^{-k\epsilon t} = 0.1$, 0.01 and $N_m = 21$, 51. Table 3 shows the values of the Bayesian confidence intervals for $p = 0.1$. For $p = 0.01$, the 80, 90, and 95% confidence intervals are all of the form $[0, N_m]$. That is, so little time was expended that essentially nothing can be said about the stock size.

The theory presented thus far was based on the uniform prior. This need not be the case; e.g. Fig. 3 shows a "tent-like" distribution in which one can assert that the region is "good" (i.e. large stock level) with probability α_3, "average" (moderate stock level) with probability α_2, and "bad" (low stock level) with probability $\alpha_1 = 1 - \alpha_2 - \alpha_3$. Thus, this theory can incorporate any intuition that a manager has about the region of interest.

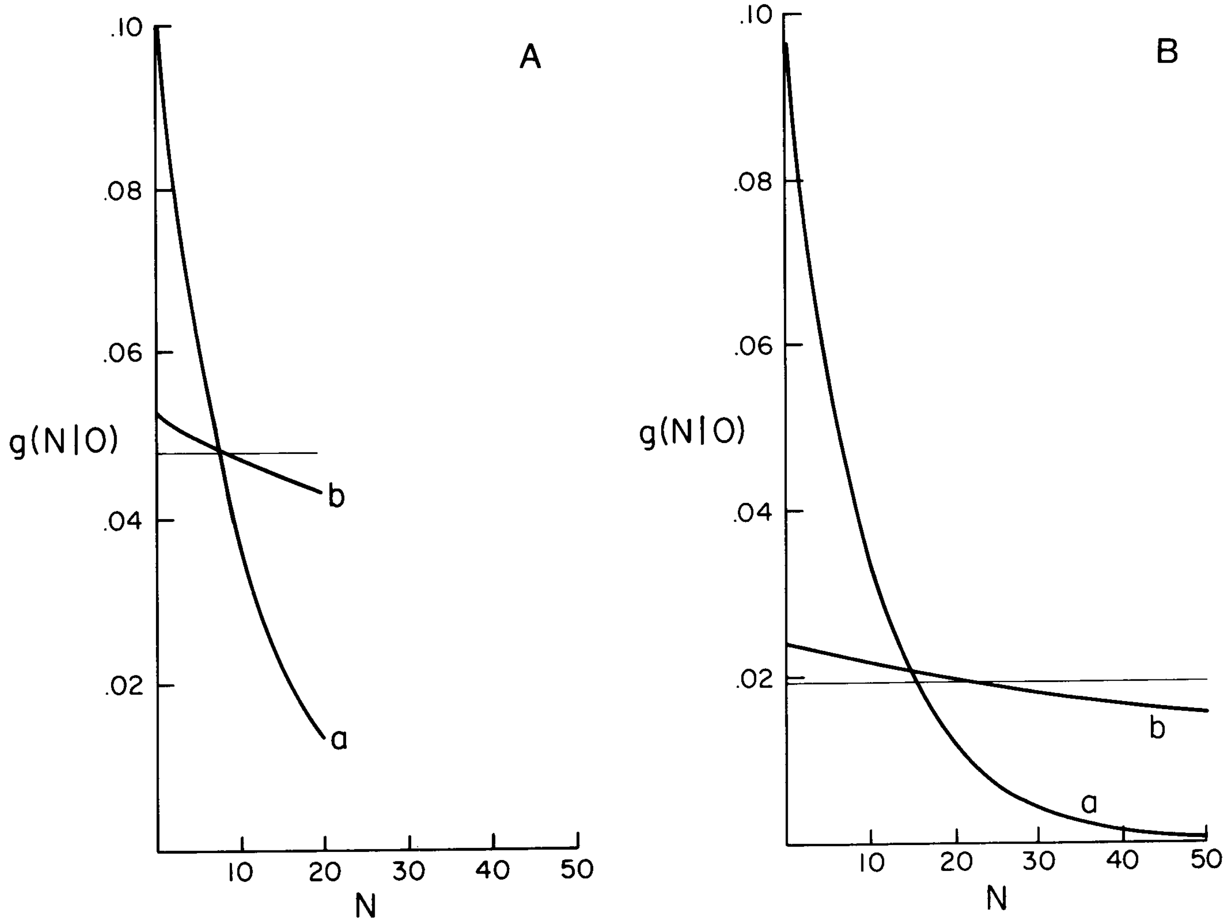

FIG. 2. Posterior distribution $g(N|0)$ when (A) $N_m = 21$ and (B) $N_m = 51$ and the probability of finding a school, given that it is present, is 0.10 (curve a) or 0.01 (curve b). The uniform prior $g_m(N)$ is also shown (straight line).

TABLE 3. Confidence intervals on stock size in the absence of catch.

	Confidence interval for		
N_m	$\gamma = 0.8$	$\gamma = 0.85$	$\gamma = 0.90$
21	[0, 15]	[0, 18]	[0, 20]
51	[0, 14]	[0, 17]	[0, 20]

4. Real Time Determination of the Length of Fishing Seasons

In this section, it is shown how the theory developed in section 2 could be used to provide a real time, adaptive determination of the length of fishing seasons. That is, imagine a situation in which the length of the season is not set in advance, but is determined as the season progresses. How should one determine the length of the rest of the season?

The basic idea is that as the stock is depleted, the time between discoveries of schools will increase. Consequently, by using the theory of section 2, one can estimate stock size from search times. The underlying assumptions are then (1) independent and random search by fishing vessels and (2) no recruitment during the season.

The ideas developed in section 2 are best illustrated by a simulation. The details of the simulation program are discussed in the Appendix. In summary, this simulated fishery lasts for up to 10 periods. The manager has the discretion, at the end of each period, to determine how much longer the fishery remains open. There are $k = 15$ vessels participating in the fishery, and $\epsilon = 0.01$. Each of three years simulated shows a pronounced decrease in catch (Fig. 4), presumably related to the depletion of the stock, but fluctuations are considerable. Table 4 shows the 25-yr averages of the catch and the coefficient of variation of the catch.

It is assumed that the manager can estimate accurately the catch in each period and the total time required to discover the fish that were set upon. This time, for instance, could be the total time at sea minus the total set time. (The catch can be estimated from landings and the search time from trip reports; see the next section.) It is also assumed that the manager operates under an exogenous biological constraint that the stock at the end of the season should be no less than a fraction f, say $f = 0.2$–0.4, of the stock at the beginning of the season. The difficult part of the decision, of course, is that the stock at the start of the season can only be estimated as data obtained throughout the season.

With this information, the following quantities are computed using the theory of section 2: (1) the maximum likelihood

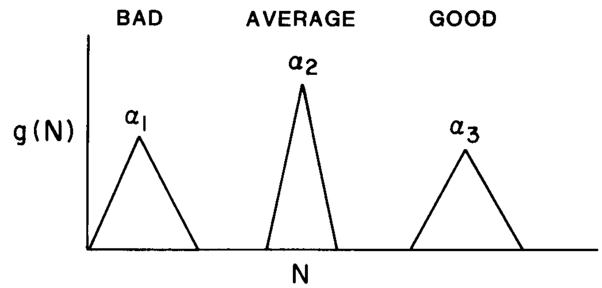

FIG. 3. Proposed prior distribution that can be used when there is a probability α_1 that the region is bad, α_2 that the region is average, and α_3 that the region is good.

estimation for the number of schools at the start of the season, $\hat{N}$, and (2) the posterior distribution for N, given the data.

For each choice of N, the expectation and variance in the catch if the season remains open for T more periods are calculated. Here, T runs between 1 and the maximum number of periods possibly remaining in the season. For a given T, the probability that the stock remains above fN if the season lasts T more periods is calculated. The calculations are then averaged over N and the results presented in tabular form. The theory for this procedure is given in equations (2.29)–(2.32).

To illustrate the results, consider the following data for the first three periods of a new season:

Period	Catch (relative units)	Search time (relative units)
1	21	1
2	11	1
3	15	1

When the theory is applied after the first period of fishing, the maximum likelihood value for the initial number of schools is $\hat{N} = 150$. Results for the rest of the season using the uniform prior (Table 5) indicate that at least one more period of fishing can be allowed without any danger to the stock. In the second period, 11 schools are caught. Thus, for the first two periods, the total catch is 32 schools. This gives $\hat{N} = 123$. The corresponding results for the rest of the season (Table 6) again indicate little chance of damage to the stock of the season if the fishery is open one more period. In the third period, 15 schools are caught; this gives 47 schools for the first three periods and $\hat{N} = 129$. The corresponding results are shown in Table 7. This procedure is repeated until either 10 periods of fishing elapse or the probability of damage to the stock becomes sufficiently high to warrant closure.

Figure 5 shows how the expected seasonal catch changes as information is gathered during the first three periods. Figures 6A and 6B show the probability that the stock level is above fN at the end of the season. These results appear to be less sensitive to the additional weekly information than the total catch is.

In a real fishery, it is likely that the data are noisier, but the methods described here can still be useful as a tool for deciding about season closures.

Other management schemes can be used with this approach. For example, many fisheries operate with a targeted escapement level. Suppose that N_s is the escapement level and that $\hat{N}$ is the MLE for N, given that n schools were encountered in search time t. Then there remain $\hat{N} - n$ schools, and a search effort of $\hat{t}$ hours will lead to the estimated expected catch $(\hat{N} - n)$

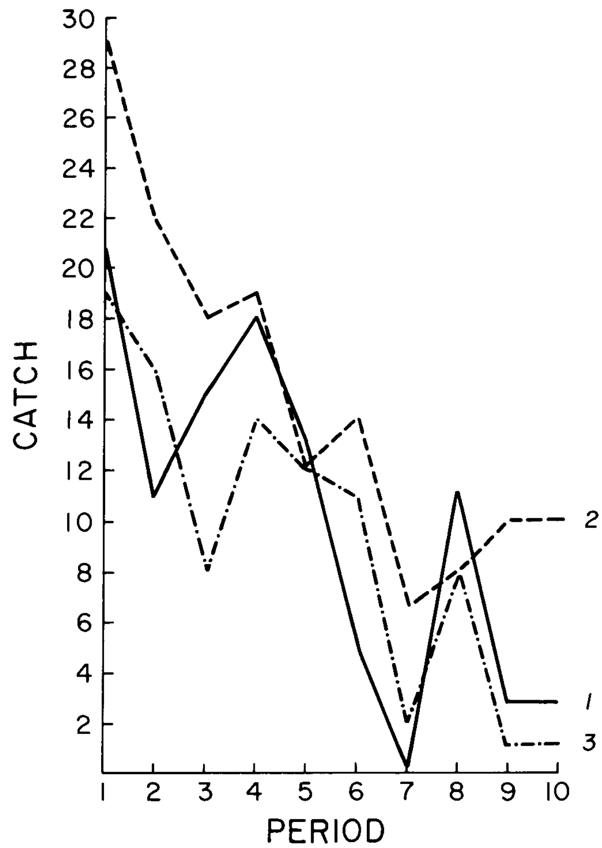

FIG. 4. Catch for three years in the simulated fishery.

TABLE 4. Historical 25-yr catch rate (relative units) for the simulated 10-period fishery.

Period	Average	Coefficient of variation
1	19.8	0.263
2	17.4	0.275
3	14.4	0.359
4	12.0	0.235
5	10.3	0.299
6	9.64	0.296
7	7.60	0.423
8	6.80	0.311
9	5.92	0.468
10	4.76	0.368

$(1 - \exp(-k\epsilon\hat{t}))$ where k is the number of vessels in the fishery. The value of $\hat{t}$ can be determined so that the targeted escapement is reached. That is, if it exists, $\hat{t}$ is the positive solution of the following equation:

$$(4.1) \qquad N_s = (\hat{N} - n)e^{-k\epsilon\hat{t}}.$$

Equation (4.1) will not have a solution if either $\hat{N} = n$ (see section 2) or $N_s \geqslant \hat{N} - n$. The latter indicates that the escapement level may not be attainable. In either case, if (4.1) does not have a positive solution, then the season should be closed immediately.

TABLE 5. Results using catch data from the first period (n = 21, t = 1, $\hat{N}$ = 150).

No. of extra periods in season	Predictions			
	Expected seasonal catch	CV in seasonal catch	Averaged Prob{stock > fN}	
			f = 0.2	f = 0.3
1	39	0.031	>0.9999	0.9994
2	55	0.039	>0.9999	0.9967
3	68	0.043	0.9997	0.9859
4	80	0.044	0.9982	0.9816
5	90	0.045	0.9921	0.8672
6	98	0.044	0.9716	0.7089
7	106	0.043	0.9175	0.4881
8	112	0.041	0.8059	0.2626
9	118	0.039	0.6283	0.0993

TABLE 6. Results using catch data from the first two periods (n = 32, t = 2, $\hat{N}$ = 132).

No. of extra periods in season	Predictions			
	Expected seasonal catch	CV in seasonal catch	Averaged Prob{stock > fN}	
			f = 0.2	f = 0.3
1	45	0.037	>0.9999	0.9989
2	56	0.046	0.9999	0.9927
3	66	0.051	0.9992	0.9664
4	73	0.053	0.9951	0.8868
5	80	0.053	0.9778	0.7198
6	87	0.052	0.9238	0.4792
7	92	0.050	0.8030	0.2426
8	96	0.049	0.6080	0.0860

A more complicated procedure for determining the remaining length of the season is to find the positive $\hat{t}(N)$ that solves

$$(4.2) \quad N_s = (N - n)e^{-k\epsilon\hat{t}}$$

and then define $\hat{t}$ by

$$(4.3) \quad \hat{t} = \sum_{N=n+1}^{\infty} g(N|n)\hat{t}(N).$$

Observe, however, that (4.1) and (4.3) give total remaining search time, and not total fishing time. One needs an algorithm that relates total fishing time to search time. Such an algorithm is described, for a particular fishery, in the next section.

5. Empirical Study of Pacific Ocean Perch (POP) near Rennell Sound, British Columbia

Rennell Sound is located on the west side of Graham Island in the Queen Charlotte Islands, British Columbia. The data on POP catch near Rennell Sound are nearly ideal for an empirical study of the usefulness of the search methods. First, the POP there is essentially a closed population, apparently the isolated remains of a once much more predominant stock (R. Stanley, Pacific Biological Station, Nanaimo, B.C., pers. comm.). There is little recruitment, so that the fishery is, in this case, exploiting an exhaustible resource. Second, the data set is sufficiently large to be broken into a subset for estimation of parameters and one for testing the methods. Third, the entire fishable area of the Sound is known (Leaman and Nagtegaal 1982). Fourth, the area was relatively unknown when first exploited. This means that the assumption of random search has considerable validity. Further discussion about the POP stock in Rennell Sound can be found in Leaman and Nagtegaal (1982), Nagtegaal et al. (1980), and Stocker (1981).

TABLE 7. Results using catch data from the first three periods (n = 47, t = 3, $\hat{N}$ = 129).

No. of extra periods in season	Expected seasonal catch	CV in seasonal catch	Averaged Prob{stock > fN}	
			f = 0.2	f = 0.3
1	59	0.038	>0.9999	0.9971
2	69	0.049	0.9998	0.9799
3	77	0.053	0.9983	0.9108
4	84	0.055	0.9886	0.7385
5	91	0.055	0.9481	0.4727
6	96	0.055	0.8372	0.2176
7	101	0.053	0.6356	0.0658

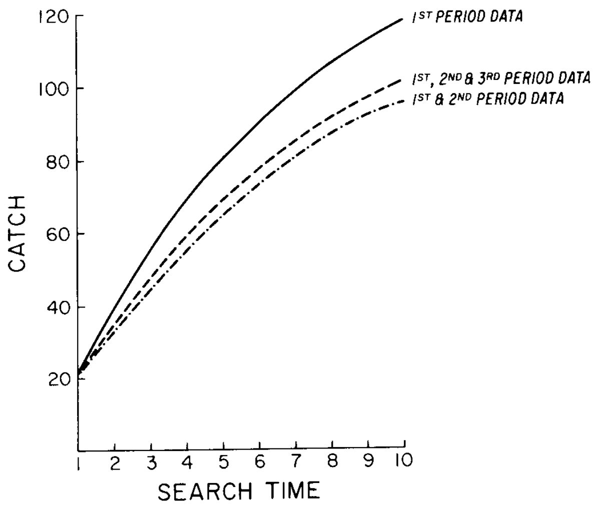

FIG. 5. Predicted catch as information is obtained.

The nature of search for POP aggregations may not fit all of the assumptions, particularly that an aggregation is fished out upon discovery. However, it will be seen that the assumptions are not violated too strongly, since the predictions based on the theory turned out to be reasonably accurate.

POP were first exploited near Rennell Sound in 1976. They are caught mainly when they form aggregations during the daylight. The log-book data kept at the Pacific Biological Station, Nanaimo, B.C., were used as a source of catch and effort information with the following rules:

(1) Each trip report was assumed to represent the encounter of one vessel with one aggregation of POP. The validity of this assumption is unknown. Given the type of log-book data available, this assumption (made for convenience) seems reasonable.

(2) The number of hours available for search was computed according to

(5.1) Search time = Days fished − set time − dark hours.

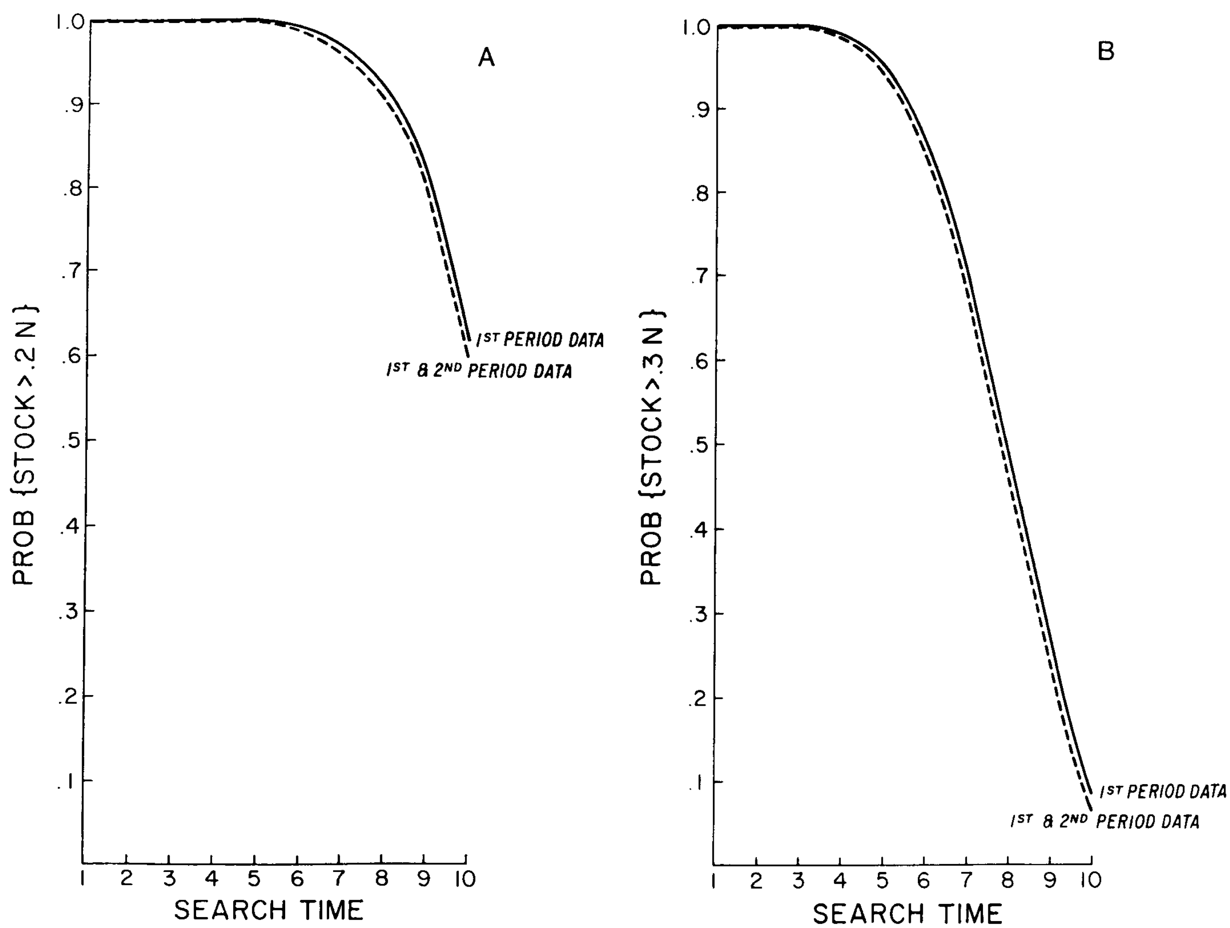

FIG. 6. Probability that the stock at the end of the season is greater than (A) 0.2 and (B) 0.3 of the initial stock.

Dark hours were computed using sunrise/sunset tables from an almanac.

By using equation (5.1) to compute search times, one implicitly assumes that fishing and searching are the only activities. If skippers logged the actual search time, then equation (5.1) would not be needed and one could find search time through dockside interviews.

For the empirical study of POP, the definition of ϵ is

$$(5.2) \quad \epsilon = \frac{vW}{A}$$

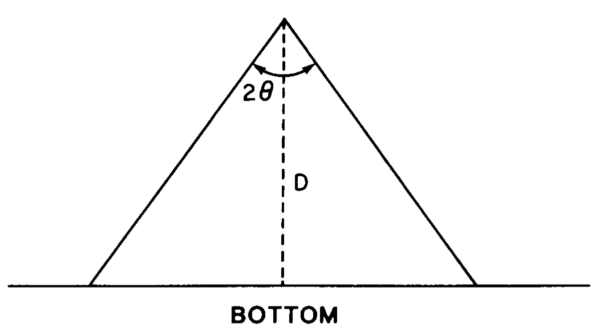

FIG. 7. Computation of the sweep width W.

where v is the speed of the vessel, W is the sweep width of the echosounder used in the search for fish, and A is the fishable area. Leaman and Nagtegaal (1982) presented data on the trawlable area in Rennell Sound as a function of interval depth. Their results are shown below:

Code	Depth interval (m)	Area (nautical miles^2)	% of total area
1	164–218	26.92	40
2	219–272	6.10	9
3	273–327	7.24	11
4	328–382	7.27	11
5	383–437	5.85	9
6	438–491	7.69	12
7	491–547	5.32	8

A different ϵ was calculated for each depth interval.

The sweep width W was calculated as follows (see Fig. 7). Imagine that the bottom is D metres below the sounder and that the sounder ensonifies an angle of 2θ below it. The ensonified width on the bottom is then

$$(5.3) \quad W = 2D \tan \theta.$$

(A more rigorous calculation, using the lateral range curve

(Koopman 1980), gives the same result, if it is assumed that the probability of detection within the triangle is 1.) The value of $\theta = 15°$ was used.

The following objectives were investigated:

(1) To use the data from January to June 1977 to estimate (a) the initial number of aggregates present, (b) confidence intervals on the initial number of aggregates, (c) biomass per aggregate, and (d) initial biomass and confidence intervals on the biomass using total biomass = (number of aggregates) × (biomass per aggregate).

(2) To predict catch as a function of effort from July 1977 through 1982, using both point estimates and confidence intervals.

(3) To investigate how the uncertainty in the estimate of biomass per aggregate affects the ultimate estimates of confidence intervals.

The data from January to June 1977 consist of 13 trip reports. For lack of any better identification procedure, it was assumed that each trip report corresponded to an encounter with one aggregation. The total search time was 181 h. The estimate for the initial number of aggregates, averaged over the seven depth intervals, is $\hat{N} = 646$ aggregates. That is, observe from (2.16) that the estimate $\hat{N}$ depends upon ϵ; one should write $\hat{N}(\epsilon)$. The ultimate estimate of $\hat{N}$ is then the average of $\hat{N}(\epsilon)$ over ϵ: $\hat{N} = E_\epsilon(\hat{N}(\epsilon))$ where E_ϵ denotes the expectation over ϵ, using the data shown below (5.2) (an incorrect method would be to use (2.16) with ϵ replaced by its average value). The average catch per encounter (i.e. per trip report) was $\hat{B}_A = 32$ tons, and the standard deviation was 22 tons (the resulting coefficient of variation is 0.69). In what follows (until the final discussion in this section), the 32 tons per aggregate will be treated as a point estimate. Finally, the average catch per set for the initial data was 2.4 tons set (with a 1-h nominal set time). It is assumed here that when a school is encountered, it is completely fished out — even if this requires more than one set. Since the theory uses search time only, there is no problem with multiple sets, as long as the appropriate corrections for times are used (as in equation (5.1)).

The estimate for the initial biomass is defined by

$$\hat{B}_0 = \hat{N}\hat{B}_A \tag{5.4}$$

where $\hat{B}_A$ is the 32 tons per aggregate and $\hat{N}$ is the averaged initial number of aggregates. Three point estimates were constructed, using the lowest, midpoint, and greatest depth for each depth interval. These results are

$$\hat{B}_0 = \begin{cases} 23\,411 \text{ tons (lowest depth)} \\ 20\,731 \text{ tons (midpoint)} \\ 18\,646 \text{ tons (greatest depth)}. \end{cases}$$

In all other calculations, only the midpoint of the depth interval was used. Note, however, that just switching the depth used in the sweep width calculation can lead to differences of 10–13% in biomass estimates. Stock surveys gave estimates of about 20 000 tons (B. Leaman, Pacific Biological Station, Nanaimo, B.C., pers. comm.).

Bayesian confidence intervals on N were constructed as described in section 2, using an improper uniform prior.

The 90% confidence intervals are

Number of aggregations: 516–776

Biomass: 16 512 – 24 832 tons.

The 95% confidence intervals are

Number of aggregations: 508–786

Biomass: 16 256 – 25 152 tons.

It should be stressed that these confidence intervals treat the biomass per aggregate as fixed. It will be seen that if one includes the uncertainty in the biomass per aggregate, then the width of the confidence interval increases considerably. It is worth noting that if the depths were aggregated, so that only one value of ϵ (averaged over the seven depth intervals) was used, the resulting estimates are 451–677 aggregates and 14 832 – 21 664 tons for the 90% interval and 443–685 aggregates and 14 176 – 21 920 tons for the 95% interval. These are about 13% too low, relative to the proper procedure of estimating $\hat{N}$ conditioned on ϵ and then averaging over ϵ.

Next, predictions were made for catches from July 1977 to 1982. The predictions were computed as follows. First, since the average size of an aggregate is 32 tons and the catch per set is 2.4 tons, there are about 13 sets per aggregate encountered, or 13 h of fishing per aggregate at the nominal value of 1 h per set. Suppose that N_R aggregates remain and that the total daylight effort will be T_E. Suppose t_s hours will be spent searching. The expected number of aggregates yet to be found is $N_R(1 - e^{-\epsilon t_s})$. The amount of time spent fishing will be 13 times this. Thus, t_s satisfies

$$t_s + 13N_R(1 - e^{-\epsilon t_s}) = T_E. \tag{5.5}$$

Solving this equation for t_s gives the search time, and from that the catch. This search time actually depends upon N_R, so that if N_R has a distribution, so does the search time. For simplicity, the search time was calculated using the maximum likelihood estimate for N_R only (initially, $N_R = \hat{N} - 13$). Bayesian confidence intervals for the predicted catch (around the expected catch of $N_R(1 - e^{-\epsilon t_s})$ aggregates × 32 tons aggregate) were constructed using a modification of the method described in section 2. Table 8 shows the point estimate for catch, the 90% confidence interval, and the actual catch observed from July 1977 onward. With the exception of 1978, the predicted and observed catches agree in the sense that the actual catch was either in the confidence interval (July–December 1977, 1980, and 1981) or very close to it (1981). The anomalous year 1978 was at least partially explained by a further investigation of the data (Mangel 1983) which shows that a number of vessels entered the Rennell Sound fishery briefly, exerted effort, obtained little or no catch, and exited. In such a situation, one expects that the predicted catch will be larger than the actual catch, which was indeed the case.

The confidence intervals in Table 8 are too small, for two reasons. First, the distribution in the number of aggregates should be reflected in a distribution on the search times. This

TABLE 8. Predicted and observed POP catches, 1977–82.

Year[a]	Effort T_E (daylight hours)	Point estimate (tons)	Catch 90% confidence interval	Actual
July–Dec. 1977	280	325	[179, 468]	430
1978	240	279	[154, 403]	119
1980	47	41	[9, 77]	51
1981	48	45	[13, 77]	43
1982	48	45	[13, 77]	79

[a]There were no substantial POP sets in 1979.

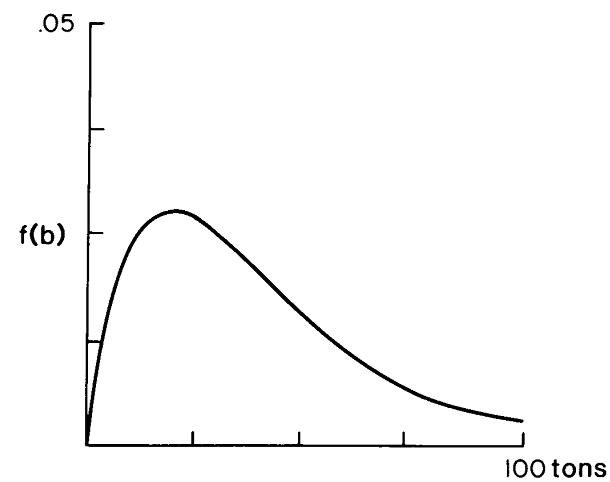

FIG. 8. Biomass density for biomass per aggregate $f(b) = 0.063^2 \exp[-0.063b]$.

calculation was not done, and only average predicted search times were used, because it is not clear that the quality of the data warrants such a calculation (this is an ex post facto prediction). In a real problem, i.e. if one were actually trying to predict future catches, the distribution on search times would need to be considered. Second, the uncertainty in the biomass per aggregate was not taken into account. In principle, the uncertainty in biomass per aggregate can be accounted for as follows. Let $f(b)$ be the probability density of B_A, the biomass per aggregate. Experience suggests that the distribution should be skewed so that the peak of the density occurs at a biomass b that is less than the mean biomass. One choice is the gamma density:

$$(5.6) \quad f(b) = \frac{\alpha^{\nu} e^{-\alpha b} b^{\nu-1}}{\Gamma(\nu)}$$

with mean ν/α and coefficient of variation $1/\sqrt{\nu}$. Matching these to 32 tons and 0.69, respectively, gives

$$(5.7) \quad \nu = 2.1, \qquad \alpha = 0.067.$$

For ease of calculation, $\nu = 2$ and $\alpha = 0.063$ will be used in the calculations that follow. Figure 8 shows the density $f(b)$. Observe how the curve is skewed to the left. Differentiation of (5.6) shows that the maximum value of $f(b)$ occurs at

$$(5.8) \quad b^* = \frac{\nu - 1}{\alpha}$$

which is 16.4 tons (exactly) or 15.9 tons (in the approximation used for calculation) per aggregate, rather than the point estimate of 32 tons.

Now the total biomass, B, is given by

$$B = \sum_{i=1}^{N} B_i,$$

where B_i is the biomass of the ith aggregate. To compute the probability density of B, an assumption must be made about the mechanism of aggregation. The two bracketing assumptions are the following ones:

(1) The biomasses of the aggregates are independent random variables with probability density given by (5.6).

(2) All aggregates are identical and have the same biomass, which is unknown. The frequency density of the unknown biomass is assumed to be given by (5.6).

If assumption (1) is correct, then the total biomass is given by the sum of N independent random variables, each of which has a gamma distribution with parameters ν and α. It is easily shown (by use of the moment generating function, which is

$$\phi(t) = \left(\frac{\alpha}{\alpha - t}\right)^{\nu}$$

for a gamma distribution with parameters ν and α) that conditioned on N, the total biomass B has a gamma density with parameters $N\nu$ and α. Consequently,

$$(5.9) \quad f_B(m)dm = \Pr\{m \leq B \leq m + dm\} = \sum_{N \geq n_s} \frac{\alpha^{N\nu} e^{-\alpha m} m^{N\nu - 1}}{\Gamma(N\nu)} g(N|n_s)dm$$

where $g(N|n_s)$ is the posterior probability that N aggregates were initially present, given that n_s aggregates were found in time t_s (here $n_s = 13$ and $t_s = 181$ h). Since the coefficient of variation of the gamma density is $1/\sqrt{\nu}$, the density (5.5) becomes more and more sharply peaked around the mean as ν increases. This means that confidence intervals based on (5.9) will be similar to those already presented. (We plan to provide a more complete discussion of confidence intervals based on (5.9) in a subsequent paper.)

If assumption (2) is correct, the density for the total biomass is given by

$$(5.10) \quad f_B(m)dm = \Pr\{m \leq B \leq m + dm\} = \Pr\{m \leq NB_A \leq m + dm\}$$

where N is the number of aggregates initially present. This equation is rewritten as

$$(5.11) \quad \Pr\{m \leq NB_A \leq m + dm\} = \sum_{N \geq n_s} \Pr\left\{\frac{m}{N} \leq B_A \leq \frac{m + dm}{N}\right\} g(N|n_s) = \sum_{N \geq n_s} f\left(\frac{m}{N}\right) g(N|n_s) \frac{dm}{N}.$$

Using the density (5.6) with parameters (5.8) in (5.11) gives

$$(5.12) \quad \Pr\{m \leq B \leq m + dm\} = \sum_{N \geq 13} (0.063)^2 \exp\left[\frac{-0.063m}{N}\right]\left(\frac{m}{N}\right) g(N|13) \frac{dm}{N}.$$

In this equation, $g(N|13)$ is given by the "negative binomial" posterior distribution (2.23) for the initial number of schools, given that $n = 13$ aggregations were found in $t = 181$ h. The distribution in (5.12) is easily computed on a desktop microcomputer. Figure 9 shows the biomass estimate computed from equation (5.12). (Actually, the calculation is somewhat more complex, since $g(N|13)$ must first be conditioned on the depth level, the biomass computed, and then averaged over all the depths.) The biomass was truncated at 52 000 tons.

The peak of the curve occurs at 9000 tons and the numerically computed mean is 19 909 tons (this mean is less than the point estimate 20 731 tons because of truncation of the density at 52 000 tons). The confidence intervals around the mean and the

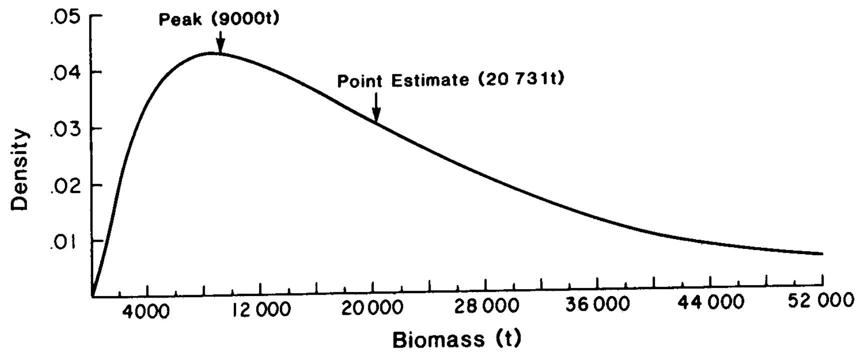

FIG. 9. Estimated biomass density function for POP under assumption (2) about aggregation.

maximum are shown below:

Confidence level	Around Mean	Around Maximum
85%	[3000, 36 000] tons	[0, 29 500] tons
90%	[1000, 38 000] tons	[0, 36 500] tons

These confidence intervals are much larger than those obtained when a point estimate of the biomass per aggregate is used. The great width of the confidence interval is caused, in large part, by the choice of the gamma distribution. That is, the long, slowly decaying tails of the gamma density lead to the long tails in the biomass distribution and thus the large confidence intervals.

One implication is clear, however, and it is that knowledge of distribution of the biomass per aggregate and the mechanism of aggregation are important if one wants meaningful confidence intervals. This point cannot be stressed too greatly. There is need for biological research to estimate the distribution of biomass per aggregate. With such large confidence intervals, a cautious management program is probably justified.

In conclusion, it is worthwhile to consider the "design of experiment" question. That is, what are the ideal data for an analysis such as this one? What data would be needed to make the analysis described here extremely accurate and as bias free as possible?

Some of the data that could be collected, but currently are not, are the following:

(1) The actual cycle of search–fish–search. If the fisherman recorded the actual search times, followed by fishing time, followed by search time, etc., then the search time would be known precisely.

(2) Fishing the same aggregation. Often the fishermen will encounter an aggregation, fish it for a while, follow it, and then fish it again. If the fishermen made a notation of such instances, it would be possible to eliminate another bias in the analyses. In another instance, schools may be partially fished by one vessel, and at some later point set upon by a different vessel. If one can estimate the probability of this occurrence and the fraction of a school taken on a set, then the theory could be easily modified accordingly (e.g. see Mangel and Clark 1983).

(3) Distribution of biomass per aggregate. If the actual distribution of biomass per aggregate were known (either from separate biomass surveys or from substantial analyses of catch data), then a calculation similar to equation (5.11) could be performed, with more confidence in the results.

6. Concluding Comments

All calculations reported in this paper were done on a desktop microcomputer. This means that the techniques are feasible for fishery managers and fishermen, who are becoming accustomed to the use of small computers in planning fishery operations. Software for the theory in section 2 is easily developed. A number of extensions of the theory presented here are currently under investigation. The first is that recruitment is not taken into account in the current theory. This is not a problem if one is interested in only a single season which is short enough so that recruitment is not significant, or the stock of interest has essentially independent generations (e.g. shrimp). However, the theory developed here can be extended to include recruitment. The second limitation is that schools of fish are typically clumped and that the theory developed here does not take clumping explicitly into account. One can do this by putting additional distributions on N or ϵ in the basic search model. This adds no additional conceptual, and very little computational, difficulty, however. A third limitation is that learning by fishermen is not taken into account. This learning has two major components. The first is local, nonrandom search. That is, after discovering a school, a vessel may execute a local, exhaustive (nonrandom) search about its find. (Tuna vessels seem to operate this way (T. Smith, NMFS-Southwest Fisheries Center, La Jolla, CA, pers. comm.).) Thus, one has two kinds of searches: random searches for "patches" (the first school found) followed by nonrandom searches within patches. The second component of learning involves learning how to fish the ground. That is, the value of A in (2.4) pertains to the entire fishing ground. As the fishermen search, they learn unprofitable areas of the ground. Thus A decreases, but the rate at which it decreases is unknown. For example, one could replace (2.4) by

$$(6.1) \quad \epsilon = \epsilon(t) = \frac{Wv}{A(t)}$$

where

$$(6.2) \quad A(t) = \frac{A_0}{1 + \theta t}$$

and A_0 is the initial size of the ground (physical size) and θ is the learning rate. If such a procedure is followed, the basic result (2.5) is replaced by

$$\text{(6.3)} \quad \Pr\{k \text{ searchers discover } n \text{ schools in } (0, t)\} = \binom{N}{n}\left[1 - \exp\left(-\frac{ktWv}{A_0} - \frac{\theta kt^2Wv}{A_0}\right)\right]^n \times \left[\exp\left(-\frac{ktWv}{A_0} - \frac{\theta kt^2Wv}{A_0}\right)\right]^{N-n}.$$

The problem now is to estimate N and θ, which is equivalent to simultaneously estimating N and p in the binomial formula

$$\binom{N}{n} p^n(1 - p)^{N-n}.$$

It is easily shown that maximum likelihood estimates give the result $\hat{N} = n$, $\hat{p} = 1$. One can however, use Bayesian methods to update a joint density for N and p. Such methods are currently being developed.

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Appendix: Description of the Simulation Program for Catch Data

The program used to generate data for the analysis in the text picks the initial number of schools, N_0, at the start of a season as follows. A random number, m, that is uniformly distributed on [100, 200] is drawn. Then N_0 is picked to be the integer part of m.

Given N_0, the catch in period l is assumed to be binomial with parameters

$$N_0 - \sum_{i=1}^{l-1} Y_i \quad \text{and} \quad 1 - e^{-k\epsilon t}.$$

Here Y_i is the observed catch in the ith period, k is the number of vessels, ϵ is search parameter defined in section 2, and t is the search time in the ith period. For simplicity, t is set equal to 1 (although t itself can be treated as a random variable (Mangel (1983)). In the simulation, the value $\epsilon = 0.01$ is used. To find the catch in period l, a random number, X, that is uniformly distributed on [0, 1] is drawn. The catch in period l, Y_l, is the integer that satisfies

$$\text{(A.1)} \quad \sum_{j=0}^{Y_l} \binom{N_0 - \sum_i^{l-1} Y_i}{j} (1 - e^{-k\epsilon t})^j \times (e^{-k\epsilon t})^{N_0 - \sum^{l-1} Y_i - j} < X < \sum_{j=0}^{Y_l+1} \binom{N_0 - \sum_i^{l-1} Y_i}{j} (1 - e^{-k\epsilon t})^j \times (e^{-k\epsilon t})^{N_0 - \sum^{l-1} Y_i - j}.$$

In the event that one of the inequalities is replaced by an equality, then Y_l is modified accordingly.