1. $a_n = \dfrac{n!}{(n-2)!}$

$$a_n = \frac{n\cdot(n-1)\cdot(n-2)!}{(n-2)!} = n(n-1)$$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n(n-1) = \infty$ DIVERGES.

2. $a_n = \dfrac{4n^3-2n}{3n^3+5}$

$\lim\limits_{n\to\infty} \dfrac{4n^3-2n}{3n^3+5}\cdot\dfrac{1/n^3}{1/n^3} = \lim\limits_{n\to\infty}\dfrac{4-2/n^2}{3+5/n^3} = \dfrac{4}{3}$ CONVERGES

3. $\displaystyle\sum_{n=1}^{\infty}\frac{(\ln n)^n}{(1+n)^n}$

Root Test

$\lim\limits_{n\to\infty}\sqrt[n]{\left|\dfrac{(\ln n)^n}{(1+n)^n}\right|} = \lim\limits_{n\to\infty}\dfrac{\ln n}{1+n} \overset{\text{L'Hospital's}}{=} \lim\limits_{n\to\infty}\dfrac{1/n}{1} = \lim\limits_{n\to\infty}\dfrac{1}{n}$

$= 0 < 1$. This series is absolutely convergent.

$\therefore$ It is convergent.

4. $\displaystyle\sum_{n=0}^{\infty}\frac{3\cdot 2^n}{5^{n+1}} = \frac{3}{5}\sum_{n=0}^{\infty}\frac{2^n}{5^n} = \frac{3}{5}\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n.$

Geometric series. Since $r = \frac{2}{5} < 1$, this series is convergent.

a is the first term of the series; $a = \frac{3}{5}\cdot\left(\frac{2}{5}\right)^1 = \frac{6}{25}$.

$$\text{SUM} = \frac{a}{1-r} = \frac{6/25}{1-2/5} = \frac{6/25}{3/5} = \frac{6}{25}\cdot\frac{5}{3} = \boxed{\frac{2}{5}}$$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{1+n^3}$.

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{1+n^3} \le \sum_{n=1}^{\infty} \frac{1}{1+n^3} \le \sum_{n=1}^{\infty} \frac{1}{n^3}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p=3>1$ so it converges.

By comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{1+n^3}$ also converges.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)5^n}{2^{2n+1}}$

Ratio Test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1+1)5^{n+1}}{2^{2(n+1)+1}}}{\frac{(n+1)5^n}{2^{2n+1}}} \right| = \lim_{n\to\infty} \left| \frac{(n+2)5^{n+1}}{2^{2n+3}} \cdot \frac{2^{2n+1}}{(n+1)5^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{5(n+2)}{4(n+1)} \right| = \frac{5}{4} > 1.$$

DIVERGES.

7. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$. Here we use integral test.

Let $f(x) = x^2 e^{-x^3}$. $f(x)$ is continuous, positive, decreasing on $[1, \infty)$.

$\displaystyle\int_1^{\infty} x^2 e^{-x^3}\,dx$. Let $u = -x^3$. Then $du = -3x^2\,dx$, $-\frac{1}{3}du = x^2 dx$

$x=1 \Rightarrow u=-1$, $x=\infty \Rightarrow u=-\infty$

$$\int_1^{\infty} x^2 e^{-x^3}\,dx = \int_{-1}^{-\infty} e^u \cdot \left(-\frac{1}{3}\right) du = \frac{1}{3}\int_{-\infty}^{-1} e^u\,du$$

Use Improper Integral here.

$$= \lim_{t\to-\infty} \frac{1}{3}\int_t^{-1} e^u\,du = \lim_{t\to-\infty} \frac{1}{3} e^u\Big]_t^{-1} = \lim_{t\to-\infty} \frac{1}{3}\left(e^{-1} - e^t\right) = \frac{1}{3e}$$

7 cont'd.

Since $\int_1^\infty x^2 e^{-x^3}\,dx$ converges to $\frac{1}{3e}$, Integral Test confirms that $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ also converges to a finite value.

8. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!\, 10^n}{3^n}$.

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!\,10^{n+1}}{3^{n+1}}}{\frac{n!\,10^n}{3^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,10^{n+1}}{3^{n+1}}\cdot\frac{3^n}{n!\,10^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)\,10}{3}\right| = \infty.$$ DIVERGES.

9. $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{5n^2+1}$.

First, consider the absolute series $\sum_{n=1}^{\infty} \frac{3n}{5n^2+1}$.

Let $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$. Use Limit Comparison Test.

$$\lim_{n\to\infty} \frac{\frac{3n}{5n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n^2}{5n^2+1} = \frac{3}{5} > 0.$$ Since this limit is finite, either both series converge or diverge.

$\sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic, i.e. diverges. $\therefore \sum_{n=1}^{\infty} \frac{3n}{5n^2+1}$ also diverges

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n 3n}{5n^2+1}$ is <u>not</u> <u>absolutely</u> <u>convergent</u>.

9 cont'd

9 Cont'd.

Now Alternating Series Test.

① $a_{n+1} \le a_n$.

two ways to show this (there could be others).

i) explicit.

$$a_{n+1} = \frac{3(n+1)}{5(n+1)^2+1}, \quad a_n = \frac{3n}{5n^2+1}$$

$$a_{n+1} = \frac{3n+3}{5n^2+10n+5+1}$$

$$a_{n+1} = \frac{3n+3}{5n^2+1+(10n+5)}, \quad a_n = \frac{3n}{5n^2+1}$$

The numerator increases by 3 while the denominator increases by $10n+5$. $\therefore$ $a_{n+1} \le a_n$.

ii) Derivative.

Let $f(x) = \dfrac{3x}{5x^2+1}$. $\quad f'(x) = \dfrac{3(5x^2+1) - 3x(10x)}{(5x^2+1)^2} = \dfrac{-15x^2+3}{(5x^2+1)^2}$

$f'(x) < 0$ for $x > 0$. Decreasing.

$\therefore$ $a_{n+1} \le a_n$.

② $$\lim_{n\to\infty} \frac{3n}{5n^2+1} = 0.$$

By Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{5n^2+1}$ converges.

$\therefore$ This series is Conditionally convergent.

1. a. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+4)^n}{n \cdot 2^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x+4)^{n+1}}{(n+1)\,2^{n+1+1}}}{\frac{(x+4)^n}{n\cdot 2^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(x+4)^{n+1}}{(n+1)\,2^{n+2}}\cdot\frac{n\cdot 2^{n+1}}{(x+4)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x+4)\cdot n}{(n+1)\cdot 2}\right| = \left|\frac{x+4}{2}\right|\cdot\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \left|\frac{x+4}{2}\right|.$$

For this series to converge, by Ratio Test, this limit must be less than 1.

Then $\left|\frac{x+4}{2}\right| < 1$.

$|x+4| < 2$. 2 is the radius of convergence, i.e. $R=2$.

$\Rightarrow$ $-2 < x+4 < 2$

$-6 < x < -2$.

Now consider the values on the edge.

i) $x=-2$.

$\displaystyle\sum_{n=1}^{\infty}\frac{(2)^n}{n\,2^{n+1}} = \sum_{n=1}^{\infty}\frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$. Harmonic, so divergent

ii) $x=-6$.

$\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n\,2^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$. Alternating Harmonic, so convergent.

$\therefore$ The interval of convergence is $[-6,-2)$

$I=[-6,-2)$, $R=2$.

1\. b. $\sum_{n=0}^{\infty} \frac{\pi^n (x-1)^{2n}}{(2n+1)!}$

Same tactic as in 1a.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\pi^{n+1}(x-1)^{2(n+1)}}{(2(n+1)+1)!}}{\frac{\pi^n (x-1)^{2n}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{\pi^{n+1}(x-1)^{2n+2}}{(2n+3)!} \cdot \frac{(2n+1)!}{\pi^n (x-1)^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\pi(x-1)^2}{(2n+3)(2n+2)} \right| = \pi(x-1)^2 \lim_{n\to\infty} \left| \frac{1}{(2n+3)(2n+2)} \right| = 0 .$$

$\therefore \quad R = \infty \quad , \quad I = (-\infty, \infty) .$

2\. $f(x) = \ln(2x+1) \quad , \quad f'(x) = \frac{2}{2x+1} .$

$\Rightarrow \frac{1}{2}\ln(2x+1) = \int \frac{1}{2x+1}\,dx .$

Series Representation for $\frac{1}{2x+1}$ ...

$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = \sum_{n=0}^{\infty} (-2x)^n .$ It is a geometric series for a fixed x, so $|-2x| < 1$ for it to be a convergent series.

$|-2x| < 1 \Rightarrow -1 < -2x < 1$

$\frac{1}{2} > x > -\frac{1}{2}, \quad |x| < \frac{1}{2} . \quad \therefore R$ (radius of convergence) $= \frac{1}{2} .$

Now, $\frac{1}{2}\ln(2x+1) = \int \frac{1}{2x+1}\,dx = \int \sum_{n=0}^{\infty} (-2x)^n\,dx = \sum_{n=0}^{\infty} \int (-2x)^n\,dx$

$$= \sum_{n=0}^{\infty} \frac{(-2)^n \cdot x^{n+1}}{n+1} + C .$$

2 Cont'd

2 Cont'd.

To find C, use $x=0$.

$$\frac{1}{2}\ln(0+1) = \sum_{n=0}^{\infty} \frac{(-2)^n \cdot 0}{n+1} + C$$

$$0 = C.$$

Then

$$\frac{1}{2}\ln(2x+1) = \sum_{n=0}^{\infty} \frac{(-2)^n x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^n \cdot x^{n+1}}{n+1}.$$

$$\Rightarrow \ln(2x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{n+1} \cdot x^{n+1}}{n+1}. = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2x)^{n+1}}{n+1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^n}{n}$$

Radius of convergence for the series for $\ln(2x+1)$ is same as for the series for $\frac{1}{1+2x}$ by the theorem in section 11.9. $\therefore$ $R = \frac{1}{2}$.

3.a. $f(x) = \tan^{-1}(3x^2)$.

We know the Maclaurin Series for $\tan^{-1}x$.

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Then

$$\tan^{-1}(3x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(3x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{3^{2n+1} \cdot x^{4n+2}}{2n+1}$$

3. b. $\int \tan^{-1}(3x^2)\,dx$

$$= \int \sum_{n=0}^{\infty} (-1)^n \cdot \frac{3^{2n+1} \cdot x^{4n+2}}{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{3^{2n+1}}{2n+1} \int x^{4n+2}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{3^{2n+1}}{2n+1} \cdot \left(\frac{x^{4n+3}}{4n+3} + C \right)$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{3^{2n+1}}{(2n+1)(4n+3)} x^{4n+3}$$

4. $f(x) = e^{-3x}$, centered at $a=2$.

Definition of a Taylor Series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f^{(n)}(x) = (-3)^n e^{-3x} \Rightarrow f^{(n)}(2) = (-3)^n e^{-6}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-3)^n e^{-6}}{n!}(x-2)^n$$

5. $f(x) = e^{2x}\cos x$

Looking for the first four non-zero terms of the Maclaurin Series.

Consider Maclaurin of e^{2x} and $\cos x$.

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + \frac{2x}{1!} + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Then

$$e^{2x}\cos x = \left(1 + \frac{2x}{1!} + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

Observing the multiplication, we will get the usual first four terms, meaning a constant, a first power of x, a second, and a third.

So we distribute each term in the first parenthesis to the second ...

$$\begin{array}{lllllll}
1 & & -\frac{x^2}{2!} & & +\frac{x^4}{4!} & & \cdots\cdots \\
 & \frac{2x}{1!} & & -\frac{2x^3}{2!} & & & \cdots\cdots \\
 & & \frac{4x^2}{2!} & & -\frac{4x^4}{2!\cdot 2!} & & \cdots\cdots \\
+ & & & \frac{8x^3}{3!} & & & \cdots\cdots \\
\hline
\end{array}$$

$$= \underbrace{1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3}_{\text{first four terms.}} + \cdots$$

Tricky Problem.

Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+2} \pi^{2n}}{2^{2n-1}(2n+1)!}$.

We observe that the Maclaurin series of $\sin x$ resembles the given series.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+2} \pi^{2n}}{2^{2n-1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3 \cdot 3^{2n+1} \cdot \pi^{2n}}{2^{2n-1}(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{\pi}{\pi} \cdot \frac{2^2}{2^2} \cdot \frac{(-1)^n \cdot 3 \cdot 3^{2n+1} \cdot \pi^{2n}}{2^{2n-1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^2 \cdot 3 \cdot 3^{2n+1} \cdot \pi^{2n+1}}{\pi \cdot 2^{2n+1}(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \left(\frac{3\pi}{2}\right)^{2n+1}}{(2n+1)!} \cdot \frac{2^2 \cdot 3}{\pi} = \frac{12}{\pi} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{3\pi}{2}\right)^{2n+1}}{(2n+1)!}$$

$$= \frac{12}{\pi} \cdot \sin\left(\frac{3\pi}{2}\right) = -\frac{12}{\pi}$$