7. find the linear approximation for $f(x) = \frac{1}{1-x}$ at $a=0$

The linear approximation for $f(x) = \frac{1}{1-x}$ at $a=0$

is given by

$L(x) = f(0) + f'(0)(x-0) = f(0) + f'(0)(x)$

where $f(0) = 1$

and $f(x) = (1-x)^{-1} \rightarrow f'(x) = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$

gives us that $f'(0) = 1$

which means

$L(x) = 1 + x$

8. find the first three derivatives of $f(x)=\tan(3x)$

given $f(x)=\tan(3x)$

By the chain Rule,

$f'(x)=(\sec^2(3x))3 = 3\sec^2(3x) = 3(\sec(3x))^2$

And again by the chain rule we have

$f''(x)= 2\cdot 3(\sec(3x))\cdot(\sec(3x)\tan(3x))3$

$= 18(\sec(3x))^2\tan(3x)$

Now we apply the product rule to obtain

$f'''(x) = 18\left((\sec(3x))^2(\tan(3x))' + ((\sec(3x))^2)'\tan(3x)\right)$

where we know by our previous calculations that

$(\tan(3x))' = 3(\sec(3x))^2$

and

$((\sec(3x))^2)' = 6(\sec(3x))^2\tan(3x)$

Therefore

$f'''(x)=18\left(\sec^2(3x)(3\sec^2(3x) + 6(\sec^2(3x)\tan(3x)\tan(3x)\right)$

$= 18(3\sec^4(3x)+6\sec^2(3x)\tan^2(3x))$

9. Assume the radius r and the Volume $V = \frac{4}{3}\pi r^3$ of a sphere are differentiable functions of time t

Express $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$

Given $V = 4/3\,\pi r^3$, with V and r differentiable functions of time we have

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(4/3\,\pi r^3\right) \rightarrow$$

$$\frac{dV}{dt} = \frac{4}{3}\pi\, 3r^2 \frac{d}{dt}(r) = \frac{4}{3}\pi 3r^2 \frac{dr}{dt}$$

which expresses $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$.

10 a) $y = x^2 + xy$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(x^2 + xy)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(xy)$$

$$\Rightarrow \frac{dy}{dx} = 2x + \frac{d}{dx}(x)\,y + x\frac{d}{dx}(y)$$

$$\Rightarrow \frac{dy}{dx} = 2x + y + x\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} - x\frac{dy}{dx} = 2x + y$$

$$\Rightarrow \frac{dy}{dx}(1-x) = 2x + y$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2x+y}{1-x}}$$

b) $x^{\frac{3}{4}} + y^{\frac{3}{4}} = 1$

$$\Rightarrow \frac{d}{dx}\left(x^{\frac{3}{4}} + y^{\frac{3}{4}}\right) = \frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}\left(x^{\frac{3}{4}}\right) + \frac{d}{dx}\left(y^{\frac{3}{4}}\right) = 0$$

con't ⇒

10 b) (con't)

$$\Rightarrow \frac{3}{4}x^{-\frac{1}{4}} + \frac{3}{4}y^{-\frac{1}{4}}\frac{d}{dx}(y) = 0$$

$$\Rightarrow \frac{3}{4}y^{-\frac{1}{4}}\frac{dy}{dx} = \frac{-3}{4}x^{-\frac{1}{4}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x^{-\frac{1}{4}}}{y^{-\frac{1}{4}}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{4}}}$$

11) Find the equation for the tangent line and the normal line to the curve given by:

$$y^2 = x^2 - x^4 \quad \text{at the point} \quad \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{4}\right)$$

Solution:

Use implicit differentiation to find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2 - x^4\right)$$

$$\Rightarrow 2y\frac{d}{dx}(y) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(-x^4\right)$$

$$\Rightarrow 2y\frac{dy}{dx} = 2x - 4x^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x - 4x^3}{2y}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right) = \frac{1 - \frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{\sqrt{3}}{3}$$

So the slope of the tangent line to the curve at $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right)$ is $\frac{\sqrt{3}}{3}$ and the slope of the normal line is $\frac{-3}{\sqrt{3}} = -\sqrt{3}$

con't

11 (con't)

Now use the point-slope formula to determine the equations of the lines

$$(y - y_0) = m(x - x_0)$$

Tangent line: $\boxed{\left(y - \frac{\sqrt{3}}{4}\right) = \frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right)}$

Normal line: $\boxed{\left(y - \frac{\sqrt{3}}{4}\right) = -\sqrt{3}\left(x - \frac{1}{2}\right)}$