# Solutions to Homework 3

November 6, 2011

**Problem 1**

Consider a production function that depends only on the number of hours of labor employed, $L$, and a technology parameter $A$: $f(L) = AL^{1/2}$

a. What is the average product of labor? What is the marginal product of labor? How does this depend on A?

$$APL = \frac{f(L)}{L} = \frac{AL^{\frac{1}{2}}}{L} = \frac{A}{L^{\frac{1}{2}}}$$

$$MPL = \frac{df}{dL} = \frac{A}{2L^{\frac{1}{2}}}$$

b. For A=1, depict in a graph the average product of labor and marginal product of labor as a function of labor hours employed. Is the average product of labor declining or increasing in labor hours employed? Is the marginal product of labor less than the average product of labor for a given level of labor employed?

c. Draw the production set and production function for A=1, A=2, and A=3.

See graph page.

**Problem 2**

For each production function below, find the marginal product of capital and labor, the marginal rate of technical substitution, and draw what the isoquant looks like for 10 units of output. Also, show whether the production function exhibits CRS, DRS, or IRS.

(a) $f(K, L) = K + 4L$

$$MPC = \frac{\delta F(K, L)}{\delta K} = 1$$

$$MPL = \frac{\delta F(K, L)}{\delta L} = 4$$

$$MRTS = \frac{MPL}{MPC} = \frac{4}{1} = 4$$

Test for returns to scale:

$$F(AK, AL) = AK + 4(AL) = AK + 4AL = A(K + 4AL) = AF(K, L)$$

This has constant returns to scale. To think of what this means, let $A = 2$. Then you have exactly doubled both your labor and capital stocks. Since $F(AK, AL) = AF(K, L)$, you have also exactly doubled output. Since output went up proportionately exactly as much as your capital and labor, we say that this production exhibits constant returns to scale.

(b) $f(K, L) = K^{\frac{1}{3}} L^{\frac{2}{3}}$

$$MPC = \frac{\delta F(K, L)}{\delta K} = \frac{1}{3} K^{-\frac{2}{3}} L^{\frac{2}{3}} = \frac{1}{3} \left( \frac{L}{K} \right)^{\frac{2}{3}}$$

$$MPL = \frac{\delta F(K, L)}{\delta L} = \frac{2}{3} K^{\frac{1}{3}} L^{-\frac{1}{3}} = \frac{2}{3} \left( \frac{L}{K} \right)^{-\frac{1}{3}}$$

$$MRTS = \frac{MPL}{MPC} = \frac{\frac{2}{3} \left( \frac{L}{K} \right)^{-\frac{1}{3}}}{\frac{1}{3} \left( \frac{L}{K} \right)^{\frac{2}{3}}} = \frac{2K}{L}$$

Test for returns to scale

$$F(AK, AL) = (AK)^{\frac{1}{3}} (AL)^{\frac{2}{3}} = A^{\frac{1}{3}} K^{\frac{1}{3}} A^{\frac{2}{3}} L^{\frac{2}{3}} = AK^{\frac{1}{3}} L^{\frac{2}{3}} = AF(K, L)$$

Constant returns to scale.

(c) $f(K, L) = K^{\frac{1}{3}} L^{\frac{1}{2}}$

$$MPC = \frac{\delta F(K, L)}{\delta K} = \frac{1}{3} K^{-\frac{2}{3}} L^{\frac{1}{2}}$$

$$MPL = \frac{\delta F(K, L)}{\delta L} = \frac{1}{2} K^{\frac{1}{3}} L^{-\frac{1}{2}}$$

$$MRTS = \frac{MPL}{MPC} = \frac{\frac{1}{2} K^{\frac{1}{3}} L^{-\frac{1}{2}}}{\frac{1}{3} K^{-\frac{2}{3}} L^{\frac{1}{2}}} = \frac{3K}{2L}$$

Test for returns to scale

$$F(AK, AL) = (AK)^{\frac{1}{3}} (AL)^{\frac{1}{2}} = A^{\frac{1}{3}} K^{\frac{1}{3}} A^{\frac{1}{2}} L^{\frac{1}{2}} = A^{\frac{5}{6}} F(K, L)$$

In this case, for example if we were to double the amounts of both capital so that $A = 2$, note that the output will only increase by $2^{\frac{5}{6}}$, which is less than two. This is an example of decreasing returns to scale.

(d) $f(K, L) = K^2 L^2$

$$MPC = \frac{\delta F(K, L)}{\delta K} = 2KL^2$$

$$MPL = \frac{\delta F(K,L)}{\delta L} = 2K^2L$$

$$MRTS = \frac{MPL}{MPC} = \frac{2K^2L}{2KL^2} = \frac{K}{L}$$

Test for returns to scale

$$F(AK, AL) = (AK)^2(AL)^2 = A^2K^2A^2L^2 = A^4K^2L^2 = A^4F(K,L)$$

If we were to double each of our inputs, we would get a proportionate increase of output of $2^4 = 16$. So doubling our inputs increases output by 16x; this is increasing returns to scale.

(e)$f(K,L) = (K^{\frac{1}{3}} + L^{\frac{1}{3}})^3$

You have to use the chain rule here:

$$\frac{dg(f(x))}{dx} = g'(f(x))f'(x)$$

$$MPC = \frac{\delta F(K,L)}{\delta K} = 3(K^{\frac{1}{3}} + L^{\frac{1}{3}})^2\left(\frac{1}{3}K^{-\frac{2}{3}}\right) = (K^{\frac{1}{3}} + L^{\frac{1}{3}})^2K^{-\frac{2}{3}}$$

$$MPL = \frac{\delta F(K,L)}{\delta L} = 3(K^{\frac{1}{3}} + L^{\frac{1}{3}})^2\left(\frac{1}{3}L^{-\frac{2}{3}}\right) = (K^{\frac{1}{3}} + L^{\frac{1}{3}})^2L^{-\frac{2}{3}}$$

$$MRTS = \frac{MPL}{MPC} = \frac{(K^{\frac{1}{3}} + L^{\frac{1}{3}})^2L^{-\frac{2}{3}}}{(K^{\frac{1}{3}} + L^{\frac{1}{3}})^2K^{-\frac{2}{3}}} = \left(\frac{K}{L}\right)^{\frac{2}{3}}$$

Returns to scale:

$$F(AK,AL) = ((AK)^{\frac{1}{3}} + (AL)^{\frac{1}{3}})^3 = (A^{\frac{1}{3}}K^{\frac{1}{3}} + A^{\frac{1}{3}}L^{\frac{1}{3}})^3 = (A^{\frac{1}{3}}(K^{\frac{1}{3}} + L^{\frac{1}{3}}))^3 = A(K^{\frac{1}{3}} + L^{\frac{1}{3}})^3 = AF(K,L)$$

CRS

**Problem 3**

A firms production function for hand-squeezed orange juice uses two inputs, oranges and labor. Let $x_1$ denote the kilograms of oranges used and let $x_2$ denote the hours of labor used in squeezing. The production function for orange juice is

$$y = x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}$$

where $y$ is measured in liters of orange juice. Let $w_1$ denote the cost of a kilogram of oranges and let $w_2$ denote the wage for an hour of labor.

(a) Find the marginal product of each input, $x_1$ and $x_2$. Find the marginal rate of technical substitution.

$$MP_1 = \frac{\delta F(x_1, x_2)}{\delta x_1} = \frac{1}{2}\left(\frac{x_2}{x_1}\right)^{\frac{1}{2}}$$

$$MP_2 = \frac{\delta F(x_1, x_2)}{\delta x_2} = \frac{1}{2}\left(\frac{x_2}{x_1}\right)^{-\frac{1}{2}}$$

$$MRTS = \frac{MP_2}{MP_1} = \frac{4}{1} = \frac{\frac{1}{2}\left(\frac{x_2}{x_1}\right)^{-\frac{1}{2}}}{\frac{1}{2}\left(\frac{x_2}{x_1}\right)^{\frac{1}{2}}} = \frac{x_1}{x_2}$$

(b) Find the choice of $x_1$ and $x_2$ that minimize the firms cost of producing $\bar{y}$ liters of orange juice. (These will be as a function of $\bar{y}$ , $w_1$, and $w_2$.)

To minimize cost, set $MRTS$ equal to relative prices:

$$MRTS = \frac{w_2}{w_1} = \frac{x_1}{x_2}$$

So

$$x_2 = \frac{w_1 x_1}{w_2}$$

If firm wants to produce $\bar{y}$ liters of OJ:

$$\bar{y} = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} = x_1^{\frac{1}{2}}\left(\frac{w_1 x_1}{w_2}\right)^{\frac{1}{2}}$$

Simplifying this:

$$x_1 = \bar{y}\left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}$$

Symmetrically,

$$x_2 = \bar{y}\left(\frac{w_1}{w_2}\right)^{\frac{1}{2}}$$

(c) Assume that the $w_1 = 3$ and $w_2 = 10$. Find the firms cost as a function of the number of liters produced.

$$C = w_1 x_2 + w_2 x_2$$

$$C(\bar{y}) = 3\bar{y}\left(\frac{10}{3}\right)^{\frac{1}{2}} + 10\bar{y}\left(\frac{3}{10}\right)^{\frac{1}{2}} = 2\sqrt{30}(\bar{y})$$

(d) What is the firms average cost of production? What is the firms marginal cost of production? Using this information, explain whether the production function exhibits IRS, DRS, or CRS.

Average cost of production equals total cost divided by quantity produced:

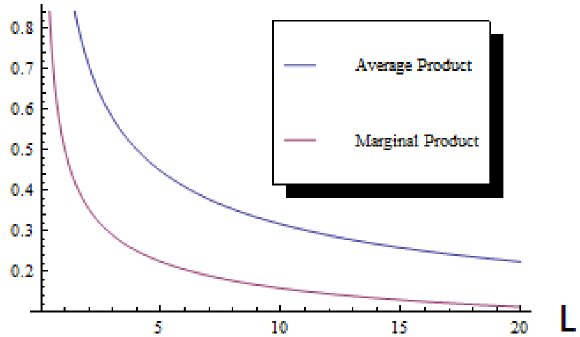
$$AC = \frac{C(\bar{y})}{\bar{y}} = \frac{2\sqrt{30}(\bar{y})}{\bar{y}} = 2\sqrt{30}$$

Marginal cost

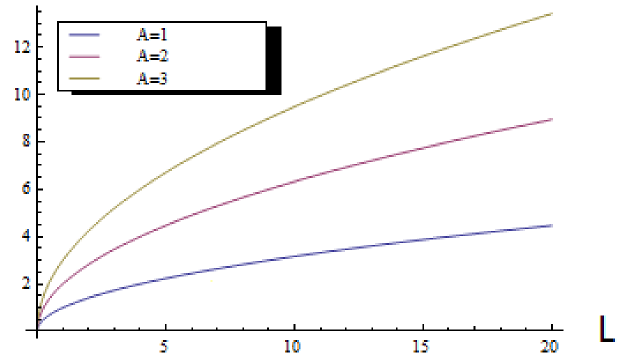
$$MC = \frac{dC}{d\bar{y}} = 2\sqrt{30}$$

Note that marginal cost is constant; this means that *marginal cost doesn't change as output or anything else changes*. Also, average cost is constant. Since costs never change, it must cost exactly the same amount for the firm to produce its 1,000,000th unit as it does to produce its first unit; this is the definition of constant returns to scale.

Question 1 (b) and (c)

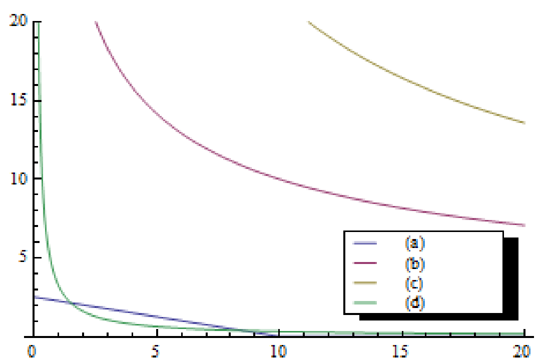



Both are decreasing in labor hours and the marginal product is less than average product (it is always equal to half of the average product)




The respective production sets include the curves and all points that are below the curves

Question 2:




These curves all represent the collection of combinations of (K,L) that will produce 10 units of output for each given production function. Note that the production function with the lowest exponents (part (c)) is the furthest away from the origin. This is because the relatively lower numbers mean more inputs must go in to produce the same amount of output