Name (please print):

1. **2.** **3.** **4.** **5.** **6.** **Total:**

Midterm Math 208 Fall 2008

Instructions: First attempt the problems closed book. Indicate how far you got closed book, then complete or check your work using any texts you choose. The exam is due at the beginning of class on Thursday, November 6.

Check with me ASAP about apparent typos, notational inconsistencies, or ambiguous definitions; the problems are pulled from several sources. If your UCSC account is not the best one to use in case of a correction email, please let me know what account to use.

1. Let $\mathcal{M}$, $\mathcal{N}$ be smooth manifolds and $f : \mathcal{M} \to \mathcal{N}$ a smooth map. Show that
   (a) $\operatorname{graph}(f) := \{(p, f(p)) \in \mathcal{M} \times \mathcal{N} : p \in \mathcal{M}\}$ is a smooth submanifold of $\mathcal{M} \times \mathcal{N}$
   (b) the canonical projection of $\operatorname{graph}(f)$ onto $\mathcal{M}$ is a diffeomorphism
   (c) $T_{(m,f(m))}(\operatorname{graph}(f)) \approx \operatorname{graph}(f_*) = \{(X, f_* X) : X \in T_p\mathcal{M}\} \subset T_p\mathcal{M} \times T_{f(m)}\mathcal{N}$
   (d) $T_{(m,f(m))}(\mathcal{M} \times \mathcal{N}) \approx T_{(m,f(m))}\operatorname{graph}(f) \oplus T_{f(m)}\mathcal{N}$ for all $p \in \mathcal{M}$

2. (a) Let $p : \mathbb{R}^k \to \mathbb{R}$ be a homogeneous polynomial of degree m, i.e.

$$p(s\,x_1, \dots, s\,x_k) = s^m p(x_1, \dots, x_k)$$

   for all $s \in \mathbb{R}$, $(x_1, \dots, x_k) \in \mathbb{R}^k$.
      i. Show that $p^{-1}(a)$ is a $k-1$ dimensional submanifold of $\mathbb{R}^k$ if $a \neq 0$.
      ii. Show that $p^{-1}(a)$ and $p^{-1}(b)$ are diffeomorphic if $a\,b > 0$.

   (b) Show that the set of 2×2 rank one matrices is a three-dimensional submanifold of $\mathbb{R}^{2\times 2}$.

3. Define $f : S^2 \to \mathbb{R}^4$ by

$$f(x, y, z) := \left(y\,z, x\,z, x\,y, x^2 + 2\,y^2 + 3\,z^2\right).$$

   (a) Show that f is an immersion.
   (b) Show that f induces an immersion $\tilde{f} : \mathbb{RP}^2 \to \mathbb{R}^4$, where $\mathbb{RP}^2$ denotes the set of lines in $\mathbb{R}^3$. (See Example 1.3 in the text if you could use more information about $\mathbb{RP}^2$.)

4. Let G be a Lie group and let G_0 denote the connected component of G containing the identity. Show that
   (a) G_0 is the only connected open subgroup of G, and
   (b) each connected component of G is diffeomorphic to G_0.

5. *Definition:* If $\pi : \mathcal{E} \to \mathcal{M}$ and $\pi' : \mathcal{E}' \to \mathcal{M}'$ are two vector bundles, a *bundle map* $F : \mathcal{E} \to \mathcal{E}'$ is a continuous map such that $\pi' \circ F = f \circ \pi$ for some continuous map $f : \mathcal{M} \to \mathcal{M}'$ and $F|_{\mathcal{E}_p} : \mathcal{E}_p \to \mathcal{E}'_{f(p)}$ is linear for each $p \in \mathcal{M}$. If F is bijective and F^{-1} is also a bundle map, F is a *bundle isomorphism.*

   (a) Find an example of a fiber-preserving diffeomorphism between vector bundles that is not a bundle isomorphism.

   (b) Show that if $\pi : \mathcal{E} \to \mathcal{M}$ and $\pi' : \mathcal{E}' \to \mathcal{M}'$ are two vector bundles and $F : \mathcal{E} \to \mathcal{E}'$ is a bijective bundle map satisfying $\pi' \circ F = \pi$, then F is a bundle isomorphism.

6. Let f be a smooth real-valued function on a smooth manifold $\mathcal{M}$. Given a smooth vector field V on $\mathcal{M}$, define the smooth function $L_V f : \mathcal{M} \to \mathbb{R}$ by $L_V f(p) := V(p)f$. Let p be a critical point of f; show that we can define the Hessian $d_p^2 f : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ of f at p as follows: given X, $Y \in T_p\mathcal{M}$, choose local smooth vector fields V and W satisfying $V(p) = X$ and $W(p) = Y$, and set $d_p^2 f(X, Y) := L_V(L_W f)(p)$.

   Hint: First show that some such local vector fields V and W exist, and then show that $L_V(L_W f)(p)$ does not depend on the choice of these vector fields, so $d_p^2 f(X, Y)$ is well-defined.