result goes back to Graves [1950]. The proof given in Box 2.5B follows Luenberger [1969].

2.5.9 Proposition *(Local Surjectivity). If $f: U \subset \mathbf{E} \to \mathbf{F}$ is C^1 and $Df(u_0)$ is onto for some $u_0 \in U$, then f is locally onto; i.e., there exist open neighborhoods U_1 of u_0 and V_1 of $f(u)$ such that $f | U_1: U_1 \to V_1$ is onto. In particular, if $Df(u)$ is onto for $u \in U$, then f is an open mapping.*

BOX 2.5B PROOF OF THE LOCAL SURJECTIVITY THEOREM

Proof. The key is to recall from Section 2.1 that $\mathbf{E}/\ker Df(u_0) = \mathbf{E}_0$ is a Banach space with norm $|||[x]||| = \inf\{\|x+u\| \mid u \in \ker Df(u_0)\}$, where $[x]$ is the equivalence class of x. To solve $f(x) = y$ we set up an iteration scheme in $\mathbf{E}_0$ and $\mathbf{E}$ simultaneously. Now $Df(u_0)$ induces an isomorphism $\mathbf{T}: \mathbf{E}_0 \to \mathbf{F}$, so $\mathbf{T}^{-1} \in \mathbf{L}(\mathbf{F}, \mathbf{E}_0)$ exists by the Banach isomorphism theorem. Write

$$f(x) = y$$

as

$$f(u_0 + h) = y, \quad \text{i.e.,} \quad \mathbf{T}^{-1}(y - f(u_0 + h)) = 0,$$

where $x = u_0 + h$.

To solve this equation, define a sequence $L_n \in \mathbf{E}/\ker Df(u_0)$ (so L_n is a coset of $\ker Df(u_0)$) and $h_n \in L_n \subset \mathbf{E}$ inductively by $L_0 = \ker Df(u_0)$, $h_0 \in L_0$ small, and

$$L_n = L_{n-1} + \mathbf{T}^{-1}(y - f(u_0 + h_{n-1})) \tag{1}$$

and selecting $h_n \in L_n$ such that

$$\|h_n - h_{n-1}\| \leqslant 2\|L_n - L_{n-1}\|. \tag{2}$$

The latter is possible since $\|L_n - L_{n-1}\| = \inf\{\|h - h_{n-1}\| \mid h \in L_n\}$.

Since $h_{n-1} \in L_{n-1}$, $L_{n-1} = \mathbf{T}^{-1}(Df(u_0)h_{n-1})$, so

$$L_n = \mathbf{T}^{-1}(y - f(u_0 + h_{n-1}) + Df(u_0)h_{n-1}).$$

Subtracting this from the same expression for L_{n-1} gives

$$L_n - L_{n-1} = -\mathbf{T}^{-1}(f(u_0 + h_{n-1}) - f(u_0 + h_{n-2}) - Df(u_0)(h_{n-1} - h_{n-2})). \tag{3}$$

For $\varepsilon > 0$ given, there is a neighborhood U of u_0 such that $\|Df(u) - Df(u_0)\| < \varepsilon$ for $u \in U$, since f is C^1. Assume inductively that $u_0 + h_{n-1} \in U$ and $u_0 + h_{n-2} \in U$. Then from the mean value inequality,

$$\|L_n - L_{n-1}\| \leqslant \varepsilon\|\mathbf{T}^{-1}\|\|h_{n-1} - h_{n-2}\|. \tag{4}$$

By (2),

$$\|h_n - h_{n-1}\| \leqslant 2\|L_n - L_{n-1}\| \leqslant 2\varepsilon\|\mathbf{T}^{-1}\|\|h_{n-1} - h_{n-2}\|.$$

Thus if ε is small,

$$\|h_n - h_{n-1}\| \leqslant \tfrac{1}{2}\|h_{n-1} - h_{n-2}\|.$$

Starting with h_0 small and $\|h_1 - h_0\| < \frac{1}{2}\|h_0\|$, $u_0 + h_n$ remains inductively in U since

$$\begin{aligned}\|h_n\| &\leqslant \|h_0\| + \|h_1 - h_0\| + \|h_2 - h_1\| + \cdots + \|h_n - h_{n-1}\| \\ &\leqslant \left(1 + \tfrac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right)\|h_0\| \leqslant 2\|h_0\|.\end{aligned}$$

It also follows that h_n is a Cauchy sequence, so it converges to some point, say h. Correspondingly, L_n converges to L and $h \in L$. Thus, from (1),

$$\mathbf{0} = \mathbf{T}^{-1}(y - f(u_0 + h))$$

and so

$$y = f(u_0 + h). \quad \blacksquare$$

This proves that for y near $y_0 = f(u_0)$, $f(x) = y$ has a solution. If there is a solution $g(y) = x$ which is C^1, then $Df(x_0) \circ Dg(y_0) = \mathbf{I}$ and so range $Dg(y_0)$ is an algebraic complement to $\ker Df(x_0)$. It follows that if range $Dg(y_0)$ is closed, then $\ker Df(x_0)$ is split.

In many applications to nonlinear partial differential equations, methods of functional analysis and elliptic operators can be used to show that $\ker Df(x_0)$ does split, even in Banach spaces. Such a splitting theorem is called the *Fredholm alternative*. For illustrations of this idea in geometry and relativity, see Fischer and Marsden [1975],[1979], and in elasticity, see Marsden and Hughes [1982, ch. 6]. For such applications, 2.5.8 suffices.

The locally injective counterpart of this theorem is the following.