Warm up example of sensitive dependence : x → 3 x (mod 1)

In[1]:=
```
triple[x_] := Mod[3 x, 1]
```

Fixed points? 0, obviously. Others?

In[3]:=
```
triple[1/2]
```

Out[3]=
$$\frac{1}{2}$$

Periodic points? A few examples:

In[2]:=
```
NestList[triple, 3/4, 2]
```

Out[2]=
$$\left\{\frac{3}{4}, \frac{1}{4}, \frac{3}{4}\right\}$$

In[5]:=
```
NestList[triple, 7/8, 2]
```

Out[5]=
$$\left\{\frac{7}{8}, \frac{5}{8}, \frac{7}{8}\right\}$$

Eventually fixed points?

In[6]:=
```
NestList[triple, 7/9, 2]
```

Out[6]=
$$\left\{\frac{7}{9}, \frac{1}{3}, 0\right\}$$

In[22]:=
```
Table[NestList[triple, 7/3^k, k], {k, 4}]
```

Out[22]=
$$\left\{\left\{\frac{7}{3}, 0\right\}, \left\{\frac{7}{9}, \frac{1}{3}, 0\right\}, \left\{\frac{7}{27}, \frac{7}{9}, \frac{1}{3}, 0\right\}, \left\{\frac{7}{81}, \frac{7}{27}, \frac{7}{9}, \frac{1}{3}, 0\right\}\right\}$$

In[19]:=
```
Table[NestList[triple, 5/3^k, k], {k, 4}]
```

Out[19]=
$$\left\{\left\{\frac{5}{3}, 0\right\}, \left\{\frac{5}{9}, \frac{2}{3}, 0\right\}, \left\{\frac{5}{27}, \frac{5}{9}, \frac{2}{3}, 0\right\}, \left\{\frac{5}{81}, \frac{5}{27}, \frac{5}{9}, \frac{2}{3}, 0\right\}\right\}$$

Eventually periodic points?

In[29]:=
```
NestList[triple, 88/5, 5]
```

Out[29]=
$$\left\{\frac{88}{5}, \frac{4}{5}, \frac{2}{5}, \frac{1}{5}, \frac{3}{5}, \frac{4}{5}\right\}$$

In[36]:=
```
NestList[triple, 39/10, 5]
```

Out[36]=
$$\left\{\frac{39}{10}, \frac{7}{10}, \frac{1}{10}, \frac{3}{10}, \frac{9}{10}, \frac{7}{10}\right\}$$

In[37]:=
```
NestList[triple, 18/43, 50]
```

Out[37]=
$$\left\{\frac{18}{43}, \frac{11}{43}, \frac{33}{43}, \frac{13}{43}, \frac{39}{43}, \frac{31}{43}, \frac{7}{43}, \frac{21}{43}, \frac{20}{43}, \frac{17}{43}, \frac{8}{43}, \frac{24}{43}, \frac{29}{43}, \frac{1}{43}, \frac{3}{43}, \frac{9}{43},\right.$$
$$\frac{27}{43}, \frac{38}{43}, \frac{28}{43}, \frac{41}{43}, \frac{37}{43}, \frac{25}{43}, \frac{32}{43}, \frac{10}{43}, \frac{30}{43}, \frac{4}{43}, \frac{12}{43}, \frac{36}{43}, \frac{22}{43}, \frac{23}{43}, \frac{26}{43}, \frac{35}{43}, \frac{19}{43},$$
$$\left.\frac{14}{43}, \frac{42}{43}, \frac{40}{43}, \frac{34}{43}, \frac{16}{43}, \frac{5}{43}, \frac{15}{43}, \frac{2}{43}, \frac{6}{43}, \frac{18}{43}, \frac{11}{43}, \frac{33}{43}, \frac{13}{43}, \frac{39}{43}, \frac{31}{43}, \frac{7}{43}, \frac{21}{43}, \frac{20}{43}\right\}$$

In[38]:=
```
Length[Union[%]]
```

Out[38]= 42

In[39]:=
```
%%[[43]]
```

Out[39]=
$$\frac{18}{43}$$

In[60]:=
```
orbit[r_] := Take[#, Length[Union[#]]] &[NestList[triple, Mod[r, 1], Denominator[r]]]
```

In[61]:=
```
orbit[1/72]
```

Out[61]=
$$\left\{\frac{1}{72}, \frac{1}{24}, \frac{1}{8}, \frac{3}{8}\right\}$$

In[49]:=
```
triple[3/8]
```

Out[49]=
$$\frac{1}{8}$$

In[62]:=
```
orbit[5/7]
```

Out[62]=
$$\left\{\frac{5}{7}, \frac{1}{7}, \frac{3}{7}, \frac{2}{7}, \frac{6}{7}, \frac{4}{7}\right\}$$

In[63]:=
```
endorbit[r_] :=
 Block[{orb = orbit[r]}, {Drop[orb, -#], Take[orb, -#]} &[Length[orbit[Last[orb]]]]]
```

In[65]:=
```
endorbit[1/72]
```

Out[65]=
$$\left\{\left\{\frac{1}{72}, \frac{1}{24}\right\}, \left\{\frac{1}{8}, \frac{3}{8}\right\}\right\}$$

In[64]:=
```
endorbit[5/7]
```

Out[64]=
$$\left\{\{\}, \left\{\frac{5}{7}, \frac{1}{7}, \frac{3}{7}, \frac{2}{7}, \frac{6}{7}, \frac{4}{7}\right\}\right\}$$

In[66]:=
```
endorbit[43/57]
```

Out[66]=
$$\left\{\left\{\frac{43}{57}\right\}, \left\{\frac{5}{19}, \frac{15}{19}, \frac{7}{19}, \frac{2}{19}, \frac{6}{19}, \frac{18}{19}, \frac{16}{19}, \frac{10}{19}, \frac{11}{19}, \frac{14}{19}, \frac{4}{19}, \frac{12}{19}, \frac{17}{19}, \frac{13}{19}, \frac{1}{19}, \frac{3}{19}, \frac{9}{19}, \frac{8}{19}\right\}\right\}$$

In[77]:=
```
FactorInteger[57]
```

Out[77]=
```
{{3, 1}, {19, 1}}
```

In[68]:=
```
endorbit[Random[Integer, {1, 100}]/Random[Integer, {1, 100}]]
```

Out[68]=
$$\left\{\{\}, \left\{\frac{8}{37}, \frac{24}{37}, \frac{35}{37}, \frac{31}{37}, \frac{19}{37}, \frac{20}{37}, \frac{23}{37}, \frac{32}{37}, \frac{22}{37}, \frac{29}{37}, \frac{13}{37}, \frac{2}{37}, \frac{6}{37}, \frac{18}{37}, \frac{17}{37}, \frac{14}{37}, \frac{5}{37}, \frac{15}{37}\right\}\right\}$$

In[69]:=
```
endorbit[Random[Integer, {1, 100}]/Random[Integer, {1, 100}]]
```

Out[69]=
$$\left\{\{\}, \left\{\frac{66}{85}, \frac{28}{85}, \frac{84}{85}, \frac{82}{85}, \frac{76}{85}, \frac{58}{85}, \frac{4}{85}, \frac{12}{85}, \frac{36}{85}, \frac{23}{85}, \frac{69}{85}, \frac{37}{85}, \frac{26}{85}, \frac{78}{85}, \frac{64}{85}, \frac{22}{85}\right\}\right\}$$

In[70]:=
```
endorbit[Random[Integer, {1, 100}]/Random[Integer, {1, 100}]]
```

Out[70]=
$$\left\{\{\}, \left\{\frac{5}{16}, \frac{15}{16}, \frac{13}{16}, \frac{7}{16}\right\}\right\}$$

In[72]:=
```
endorbit[Random[Integer, {1, 100}]/Random[Integer, {1, 100}]]
```

Out[72]=
$$\left\{\{\}, \left\{\frac{1}{10}, \frac{3}{10}, \frac{9}{10}, \frac{7}{10}\right\}\right\}$$

Sensitive dependence?

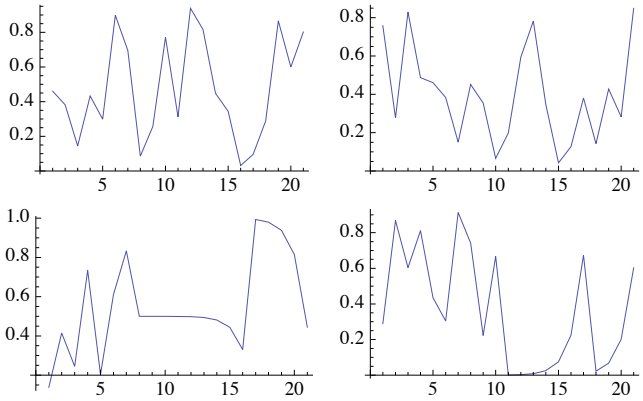
In[82]:=
```
TableForm[NestList[triple, Array[Random[] &, 4], 20]]
```

Out[82]//TableForm=

0.460487	0.75878	0.138317	0.289319
0.381462	0.27634	0.414952	0.867957
0.144385	0.82902	0.244856	0.603872
0.433156	0.487059	0.734568	0.811615
0.299469	0.461177	0.203703	0.434844
0.898407	0.383531	0.611108	0.304531
0.695221	0.150593	0.833325	0.913592
0.0856626	0.45178	0.499974	0.740775
0.256988	0.355339	0.499923	0.222325
0.770963	0.0660178	0.499768	0.666975
0.312889	0.198053	0.499305	0.000924707
0.938668	0.59416	0.497914	0.00277412
0.816004	0.78248	0.493743	0.00832237
0.448012	0.347439	0.481228	0.0249671
0.344035	0.042317	0.443683	0.0749013
0.032105	0.126951	0.331048	0.224704
0.0963151	0.380853	0.993144	0.674112
0.288945	0.14256	0.979431	0.0223351
0.866836	0.427679	0.938293	0.0670053
0.600507	0.283036	0.814879	0.201016
0.801521	0.849109	0.444638	0.603048

In[83]:=
```
GraphicsArray[Partition[ListPlot[#, PlotJoined → True] & /@ Transpose[%], 2]]
```

Out[83]=



In[89]:=
```
Random[] + Table[1/10^(6-j), {j, 4}]
```

Out[89]=
```
{0.40376, 0.40385, 0.40475, 0.41375}
```

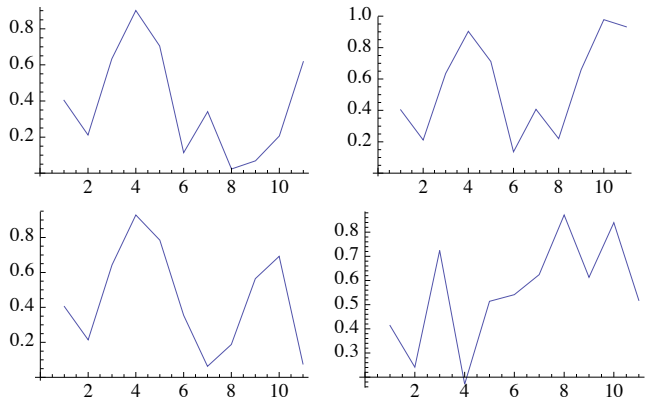
In[90]:=
```
TableForm[NestList[triple, %, 10]]
```

Out[90]//TableForm=

0.40376	0.40385	0.40475	0.41375
0.21128	0.21155	0.21425	0.24125
0.633839	0.634649	0.642749	0.723749
0.901517	0.903947	0.928247	0.171247
0.704552	0.711842	0.784742	0.513742
0.113655	0.135525	0.354225	0.541225
0.340965	0.406575	0.0626746	0.623675
0.0228938	0.219724	0.188024	0.871024
0.0686813	0.659171	0.564071	0.613071
0.206044	0.977514	0.692214	0.839214
0.618132	0.932542	0.0766421	0.517642

In[91]:=
```
GraphicsArray[Partition[ListPlot[#, PlotJoined → True] & /@ Transpose[%], 2]]
```

Out[91]=



In[97]:=
```
Random[] + Table[1/10^(10-j), {j, 4}]
```

Out[97]=
```
{0.683795, 0.683795, 0.683795, 0.683796}
```

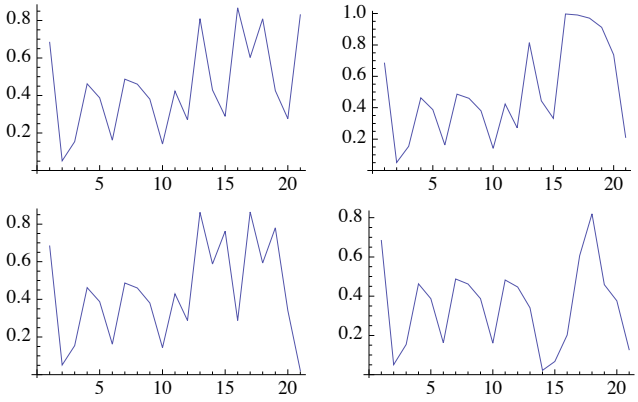
In[98]:=
```
TableForm[NestList[triple, %, 20]]
```

Out[98]//TableForm=

0.683795	0.683795	0.683795	0.683796
0.0513856	0.0513857	0.0513859	0.0513886
0.154157	0.154157	0.154158	0.154166
0.462471	0.462471	0.462473	0.462498
0.387412	0.387413	0.38742	0.387493
0.162236	0.162238	0.16226	0.162479
0.486708	0.486714	0.48678	0.487436
0.460123	0.460143	0.46034	0.462308
0.380369	0.380428	0.381019	0.386924
0.141108	0.141285	0.143056	0.160771
0.423324	0.423855	0.429169	0.482313
0.269971	0.271565	0.287508	0.44694
0.809912	0.814695	0.862524	0.340821
0.429735	0.444084	0.587573	0.0224635
0.289204	0.332251	0.762718	0.0673904
0.867613	0.996753	0.288155	0.202171
0.602839	0.99026	0.864465	0.606513
0.808518	0.970779	0.593394	0.81954
0.425553	0.912337	0.780181	0.458621
0.276659	0.737012	0.340544	0.375864
0.829976	0.211035	0.021631	0.127592

In[99]:=
```
GraphicsArray[Partition[ListPlot[#, PlotJoined → True] & /@ Transpose[%], 2]]
```

Out[99]=



Back to the logistic map : x → 4 x (mod 1)

■ Definitions

In[100]:=
```
f4[x_] := 4 x (1 - x)
```

In[101]:=
```
symbols[x_, k_] := StringJoin[If[# < .5, "L", "R"] & /@ NestList[f4, x, k]]
```

In[103]:=
```
rootsf4[k_] := Union[N[x /. Solve[Nest[f4, x, k] == 0, x]], SameTest → (Abs[#1 - #2] < 10^-14 &)]
```

In[104]:=
```
ints[k_] :=
  TableForm[Drop[Join @@ Table[{symbols[1/2 (#[[j]] + #[[j + 1]]), k], #[[j + 1]]}, {j, Length[#] - 1}] &[
     rootsf4[k + 2]], -1]]
```

In[105]:=
```
howclose[k_] := {Min[#], Max[#], N[π/2^(k+2)]} &[Drop[#, 1] - Drop[#, -1] &[rootsf4[k + 2]]]
```

■ Sample runs

In[102]:=
```
symbols[.3, 20]
```

Out[102]=
```
LRRRLLLRLRLLRRLLRRLLR
```

In[106]:=
```
ints[1]
```

Out[106]//TableForm=

LL	
0.146447	
LR	
0.5	
RR	
0.853553	
RL	

In[107]:=
```
ints[2]
```

Out[107]//TableForm=

LLL	
0.0380602	
LLR	
0.146447	
LRR	
0.308658	
LRL	
0.5	
RRL	
0.691342	
RRR	
0.853553	
RLR	
0.96194	
RLL	

In[108]:= **ints[3]**

Out[108]//TableForm=

LLLL
0.00960736
LLLR
0.0380602
LLRR
0.0842652
LLRL
0.146447
LRRL
0.222215
LRRR
0.308658
LRLR
0.402455
LRLL
0.5
RRLL
0.597545
RRLR
0.691342
RRRR
0.777785
RRRL
0.853553
RLRL
0.915735
RLRR
0.96194
RLLR
0.990393
RLLL

In[109]:= **TableForm[Array[howclose, 6]]**

Out[109]//TableForm=

0.146447	0.353553	0.392699
0.0380602	0.191342	0.19635
0.00960736	0.0975452	0.0981748
0.00240764	0.0490086	0.0490874
0.000602272	0.0245338	0.0245437
0.000150591	0.0122706	0.0122718