

■ **Definitions**

■ **Roots of iterations of logistic maps**

$$\text{Nest}\left[a \# (1 - \#) \ \&, \frac{1}{2} \left(1 + \sqrt{1 - \frac{4}{a}}\right), 2\right] // \text{Factor}$$

0

$$\text{Nest}\left[a \# (1 - \#) \ \&, \frac{1}{2} \left(1 - \sqrt{1 - \frac{4}{a}}\right), 2\right] // \text{Factor}$$

0

In[708]:= $\text{Table}\left[\frac{\text{Nest}[a \# (1 - \#) \ \&, x, j + 1]}{a \text{ Nest}[a \# (1 - \#) \ \&, x, j]}, \{j, 5\}\right] // \text{Factor}$

Factor[% /. a -> 4] // TableForm

$$\begin{aligned} &(-1 + 2x)^2 \\ &(1 - 8x + 8x^2)^2 \\ &(1 - 32x + 160x^2 - 256x^3 + 128x^4)^2 \\ &(1 - 128x + 2688x^2 - 21504x^3 + 84480x^4 - 180224x^5 + 212992x^6 - 131072x^7 + 32768x^8)^2 \\ &(1 - 512x + 43520x^2 - 1462272x^3 + 25798656x^4 - 275185664x^5 + 1926299648x^6 - 9313976320x^7 + 321332 \end{aligned}$$

Factor[x /. Solve[# == 0, x] & /@ ((# /. x_ -> x) & /@ %)] // TableForm

$$\begin{array}{cccc} \frac{1}{2} & & & \\ \frac{1}{4} (2 - \sqrt{2}) & \frac{1}{4} (2 + \sqrt{2}) & & \\ \frac{1}{4} (2 - \sqrt{2 - \sqrt{2}}) & \frac{1}{4} (2 + \sqrt{2 - \sqrt{2}}) & \frac{1}{4} (2 - \sqrt{2 + \sqrt{2}}) & \frac{1}{4} (2 + \sqrt{2 + \sqrt{2}}) \\ \frac{1}{4} \left(2 - \sqrt{2 - \sqrt{2 - \sqrt{2}}}\right) & \frac{1}{4} \left(2 + \sqrt{2 - \sqrt{2 - \sqrt{2}}}\right) & \frac{1}{4} \left(2 - \sqrt{2 + \sqrt{2 - \sqrt{2}}}\right) & \frac{1}{4} \left(2 + \sqrt{2 + \sqrt{2 - \sqrt{2}}}\right) \\ \frac{1}{4} \left(2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2}}}}\right) & \frac{1}{4} \left(2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2}}}}\right) & \frac{1}{4} \left(2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2}}}}\right) & \frac{1}{4} \left(2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2}}}}\right) \end{array}$$

In[710]:= $\text{Solve}[\# == 0, x] \ \& \ [x^2 + 2]$

Out[710]= $\left\{\left\{x \rightarrow -i \sqrt{2}\right\}, \left\{x \rightarrow i \sqrt{2}\right\}\right\}$

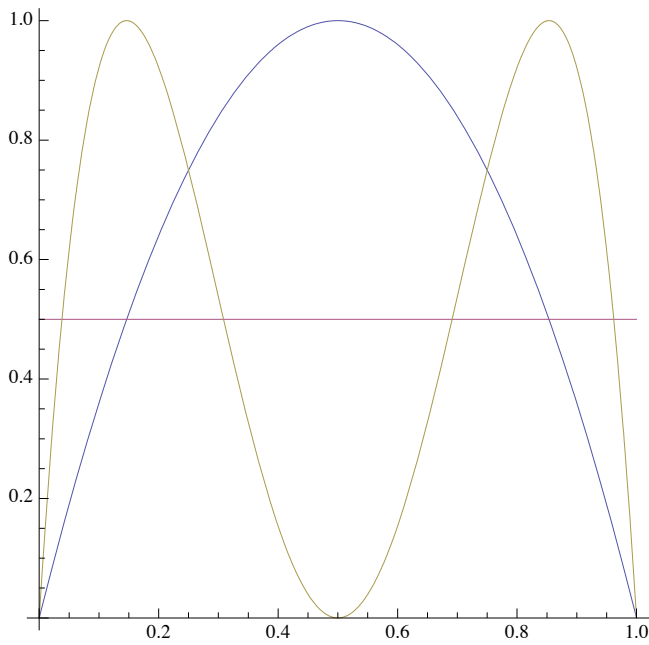
Sort /@N[%] // TableForm

0.5							
0.146447	0.853553						
0.0380602	0.308658	0.691342	0.96194				
0.00960736	0.0842652	0.222215	0.402455	0.597545	0.777785	0.915735	0.
0.00240764	0.0215298	0.0590394	0.113495	0.182803	0.264302	0.354858	0.

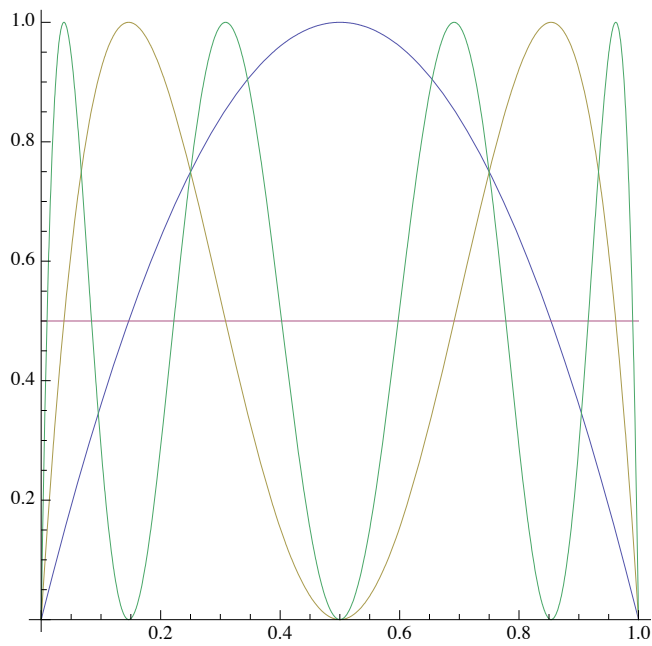
f4[%] // TableForm

1.							
0.5	0.5						
0.146447	0.853553	0.853553	0.146447				
0.0380602	0.308658	0.691342	0.96194	0.96194	0.691342	0.308658	0.0
0.00960736	0.0842652	0.222215	0.402455	0.597545	0.777785	0.915735	0.9

Plot[{f4[x], 1/2, f4[f4[x]]}, {x, 0, 1}, AspectRatio -> 1]

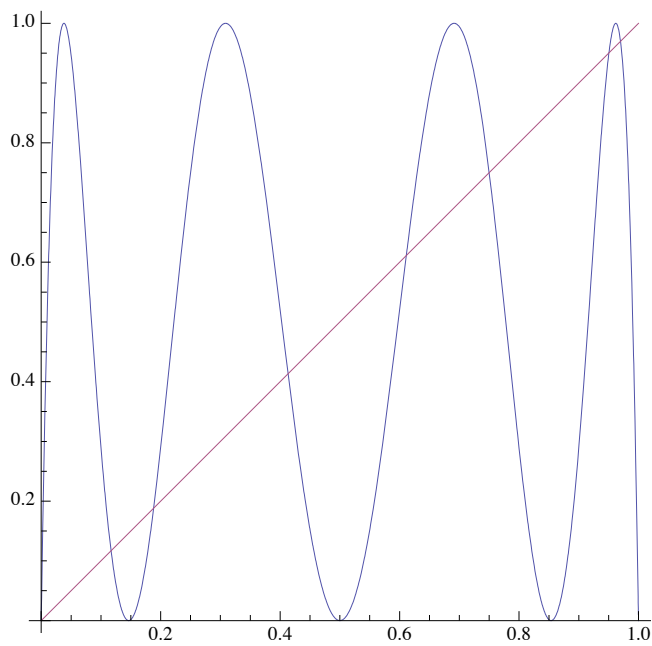


```
Plot[{f4[x], 1/2, f4[f4[x]], f4[f4[f4[x]]]}, {x, 0, 1}, AspectRatio -> 1]
```



■ Periodic points of f4

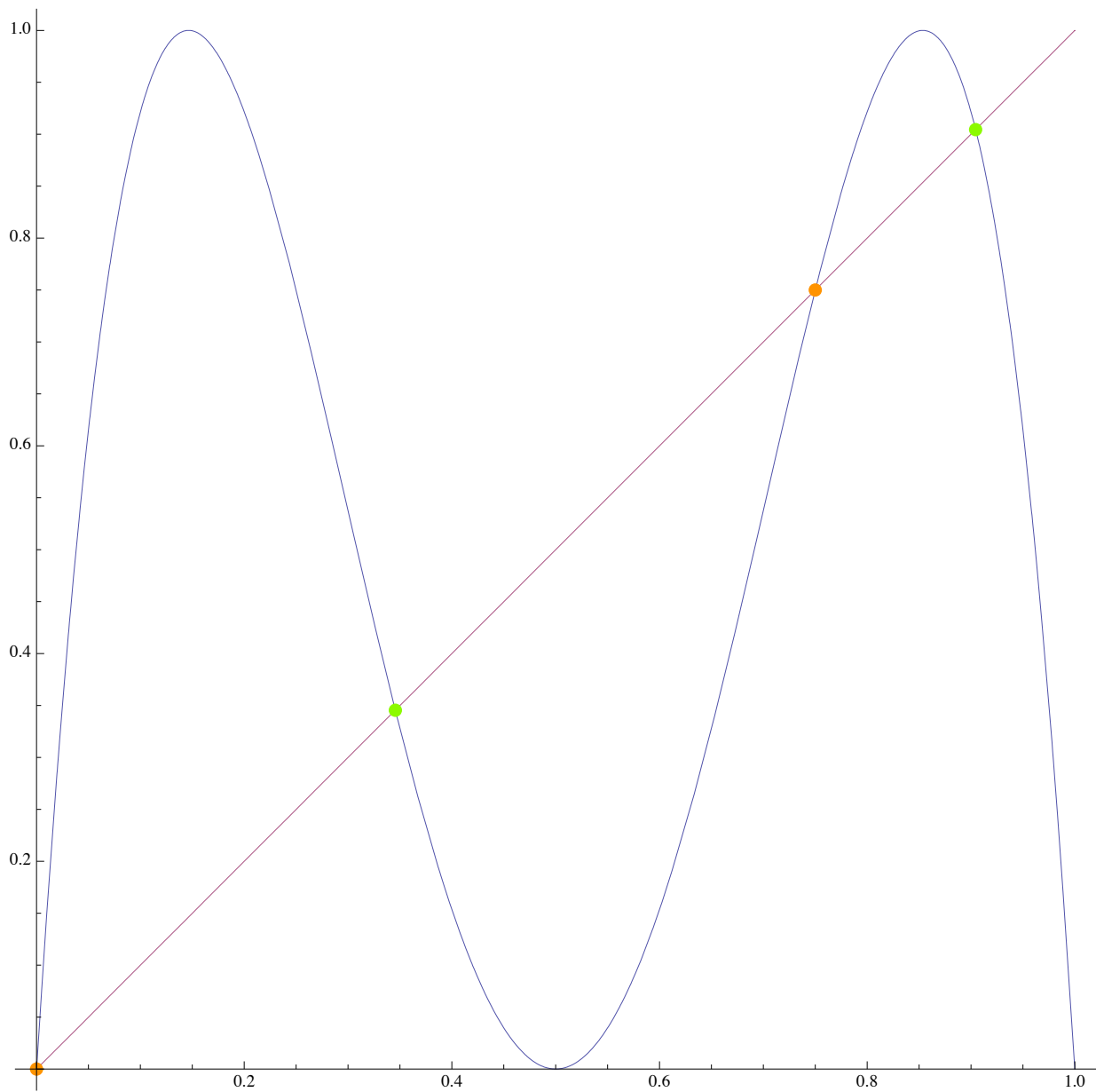
```
Plot[{Nest[f4, x, 3], x}, {x, 0, 1}, AspectRatio -> 1]
```



```
x /. Solve[f4[f4[x]] == x, x]
```

$$\left\{0, \frac{3}{4}, \frac{1}{8} (5 - \sqrt{5}), \frac{1}{8} (5 + \sqrt{5})\right\}$$

showppf4[2]



x /. Solve[f4[f4[f4[x]]] == x, x]

$$\left\{0, \frac{3}{4}, \frac{1}{2} - \frac{1}{8} (1 - i \sqrt{3}) \left(\frac{1}{2} (1 + i \sqrt{3})\right)^{1/3} - \frac{1}{4} \left(\frac{1}{2} (1 + i \sqrt{3})\right)^{2/3}, \right.$$

$$\frac{1}{2} + \frac{1}{4} \left(\frac{1}{2} (1 + i \sqrt{3})\right)^{1/3} + \frac{1}{2 \cdot 2^{2/3} (1 + i \sqrt{3})^{1/3}}, \frac{1}{2} - \frac{1 - i \sqrt{3}}{4 \cdot 2^{2/3} (1 + i \sqrt{3})^{1/3}} - \frac{(1 + i \sqrt{3})^{4/3}}{8 \cdot 2^{1/3}},$$

$$\frac{7}{12} + \frac{7^{2/3}}{6 \cdot 2^{2/3} (-1 + 3 i \sqrt{3})^{1/3}} + \frac{1}{12} \left(\frac{7}{2} (-1 + 3 i \sqrt{3})\right)^{1/3},$$

$$\frac{7}{12} - \frac{\left(\frac{7}{2}\right)^{2/3} (1 + i \sqrt{3})}{12 (-1 + 3 i \sqrt{3})^{1/3}} - \frac{1}{24} (1 - i \sqrt{3}) \left(\frac{7}{2} (-1 + 3 i \sqrt{3})\right)^{1/3},$$

$$\left. \frac{7}{12} - \frac{\left(\frac{7}{2}\right)^{2/3} (1 - i \sqrt{3})}{12 (-1 + 3 i \sqrt{3})^{1/3}} - \frac{1}{24} (1 + i \sqrt{3}) \left(\frac{7}{2} (-1 + 3 i \sqrt{3})\right)^{1/3} \right\}$$

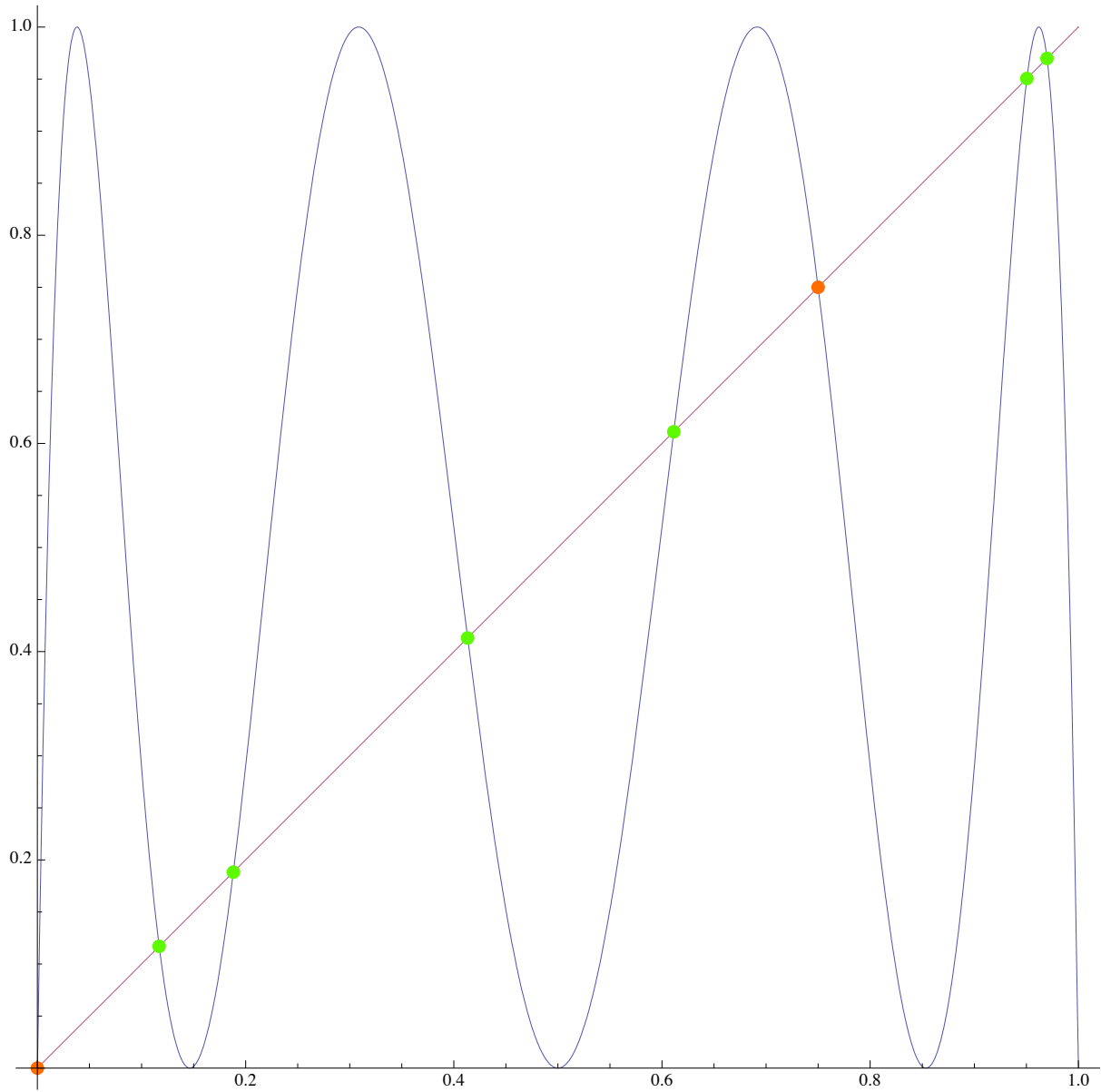
N[%]

$$\{0., 0.75, 0.116978 + 2.77556 \times 10^{-17} i, 0.969846 - 1.38778 \times 10^{-17} i, 0.413176 + 2.77556 \times 10^{-17} i, \\ 0.950484 + 1.38778 \times 10^{-17} i, 0.188255 + 0. i, 0.61126 - 2.77556 \times 10^{-17} i\}$$

Sort[Chop[%]]

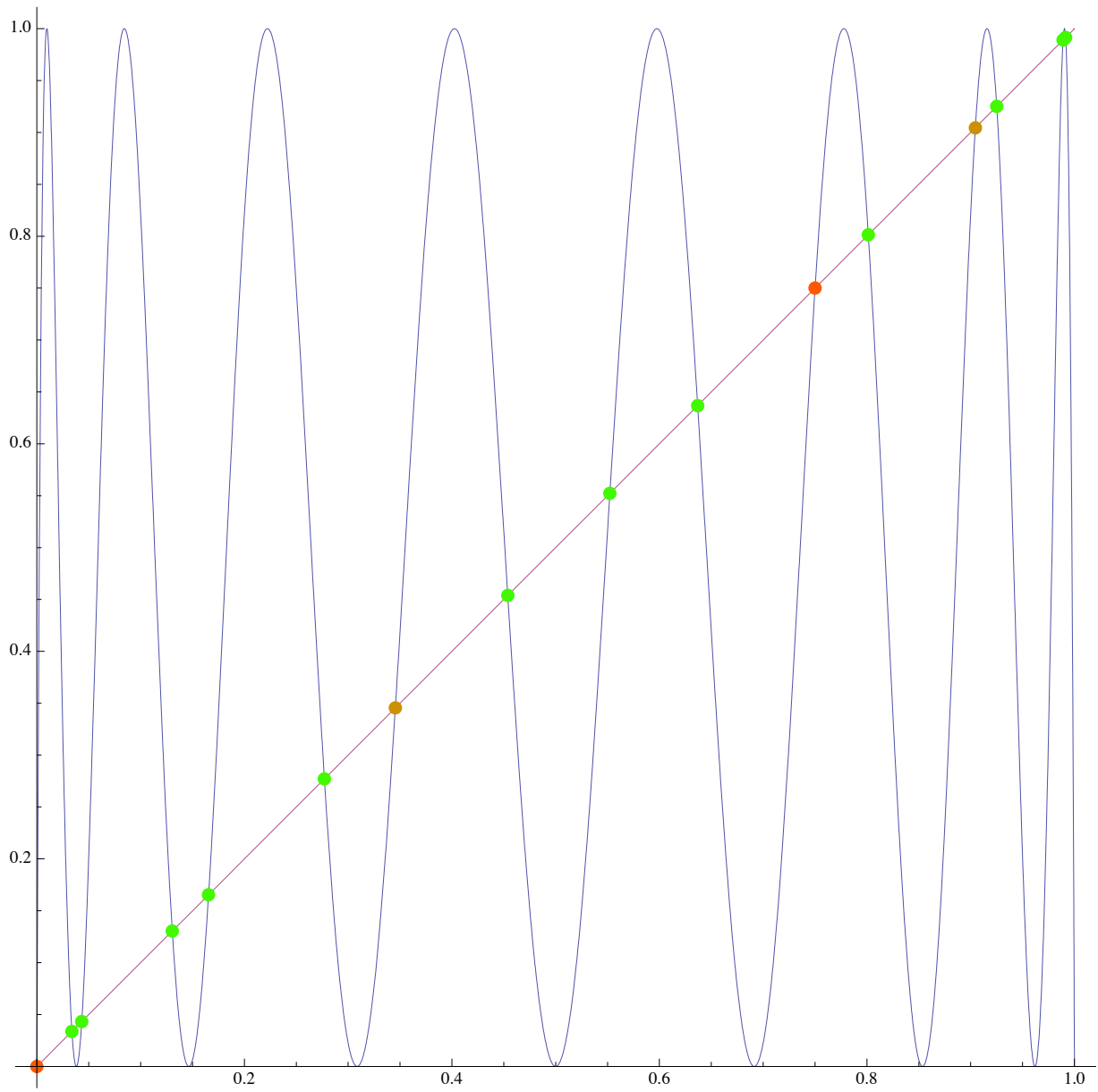
$$\{0, 0.116978, 0.188255, 0.413176, 0.61126, 0.75, 0.950484, 0.969846\}$$

showppf4[3]

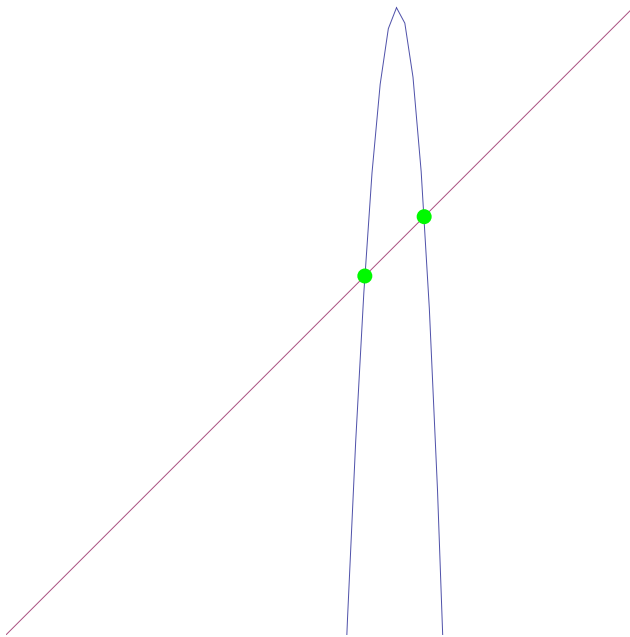


`p4 = x /. Solve[Nest[f4, x, 4] == x, x] // N // Chop`

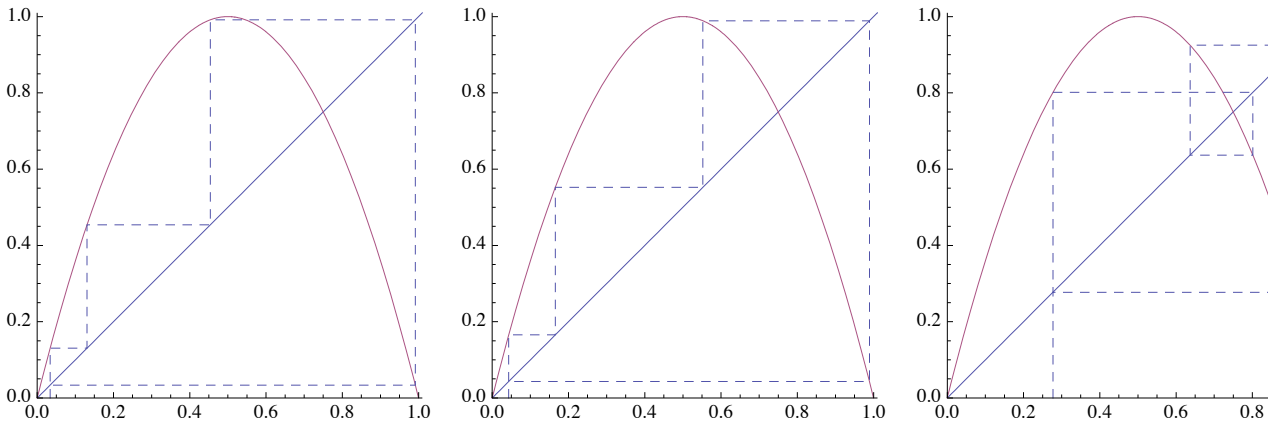
```
{0, 0.75, 0.345492, 0.904508, 0.0432273, 0.552264, 0.165435, 0.989074,  
0.0337639, 0.130496, 0.277131, 0.453866, 0.636831, 0.801317, 0.925109, 0.991487}
```

showppf4[4]

```
Show[%, PlotRange -> {{.975, 1}, {.975, 1}}]
```

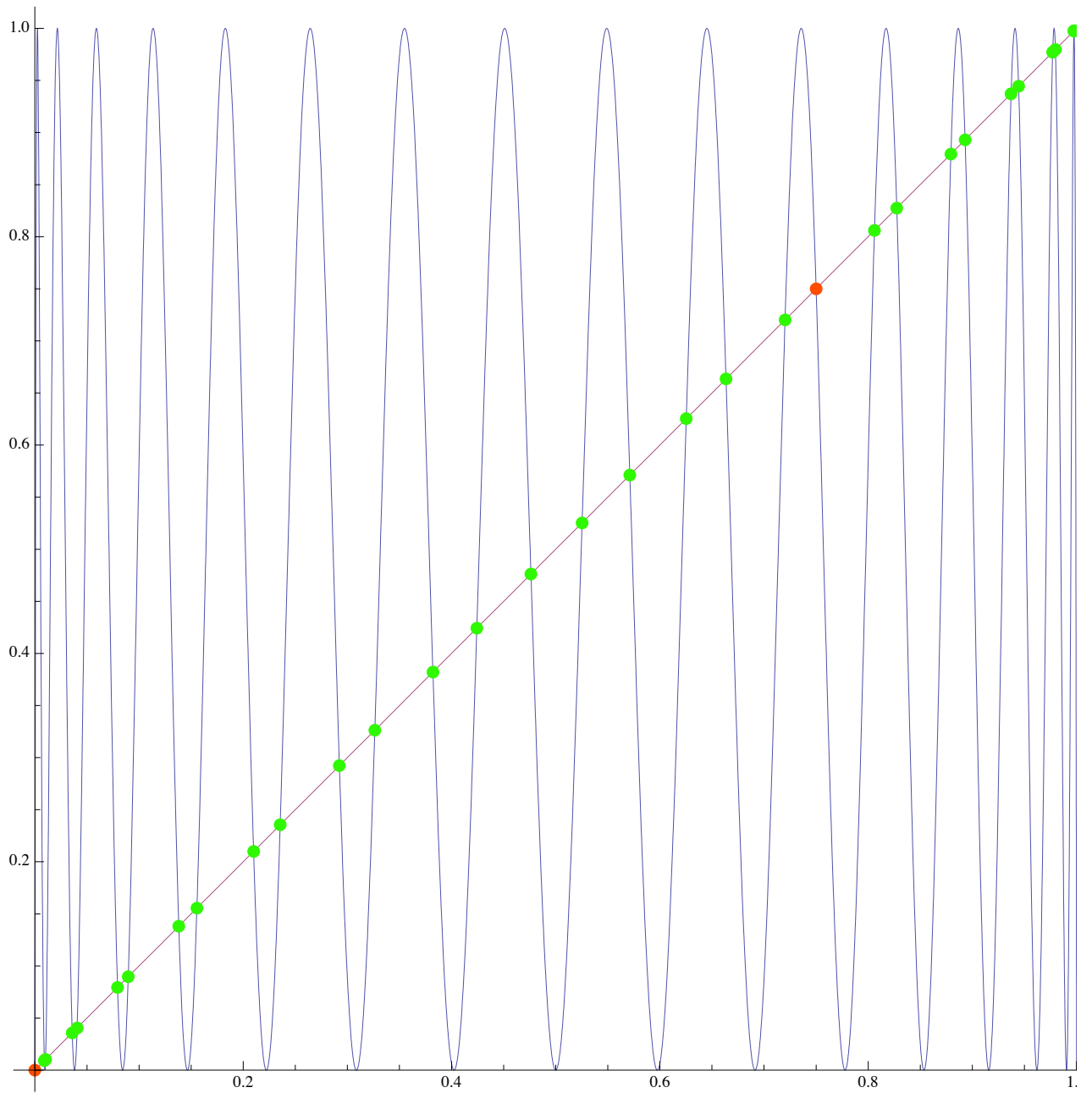


```
GraphicsArray[cobweb[f4, #, {0, 1.01}, 4] & /@distinctorbits[Drop[p4, 4], 4]]
```

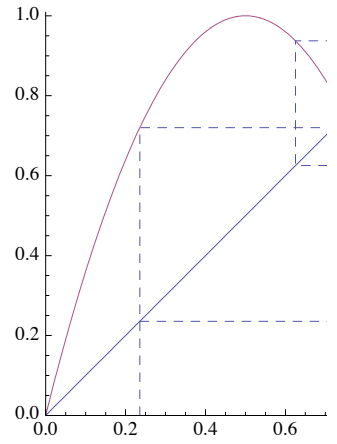
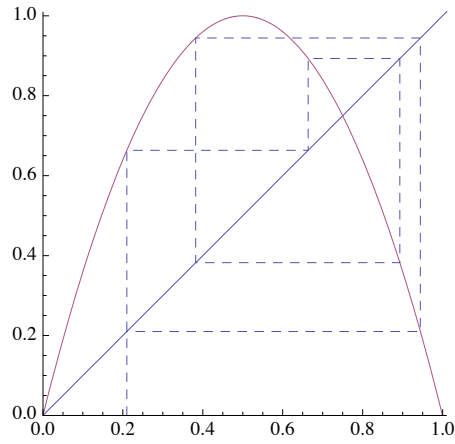
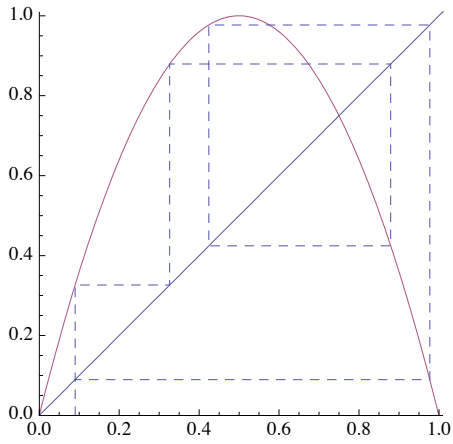
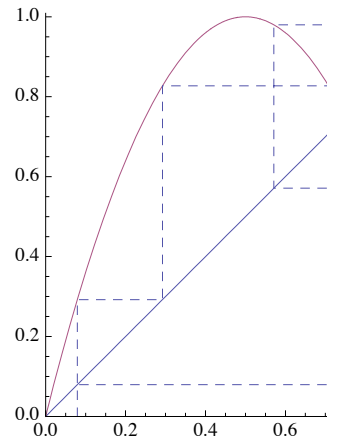
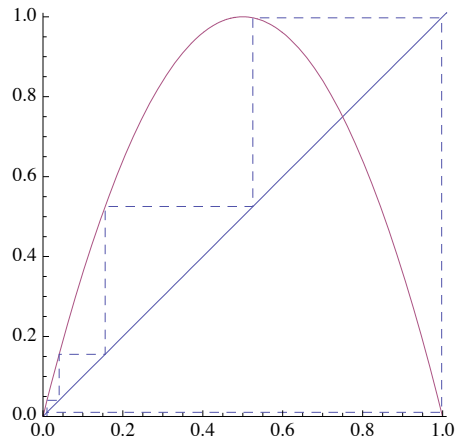
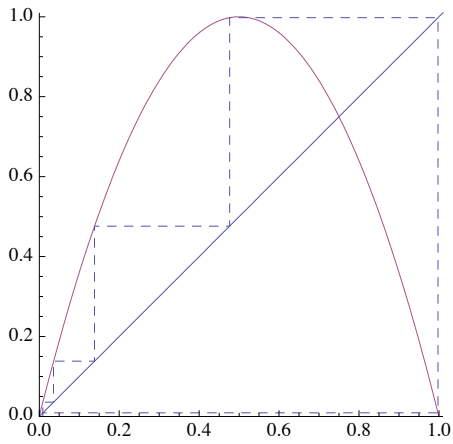


```
p5 = x /. Solve[Nest[f4, x, 5] == x, x] // N // Chop
```

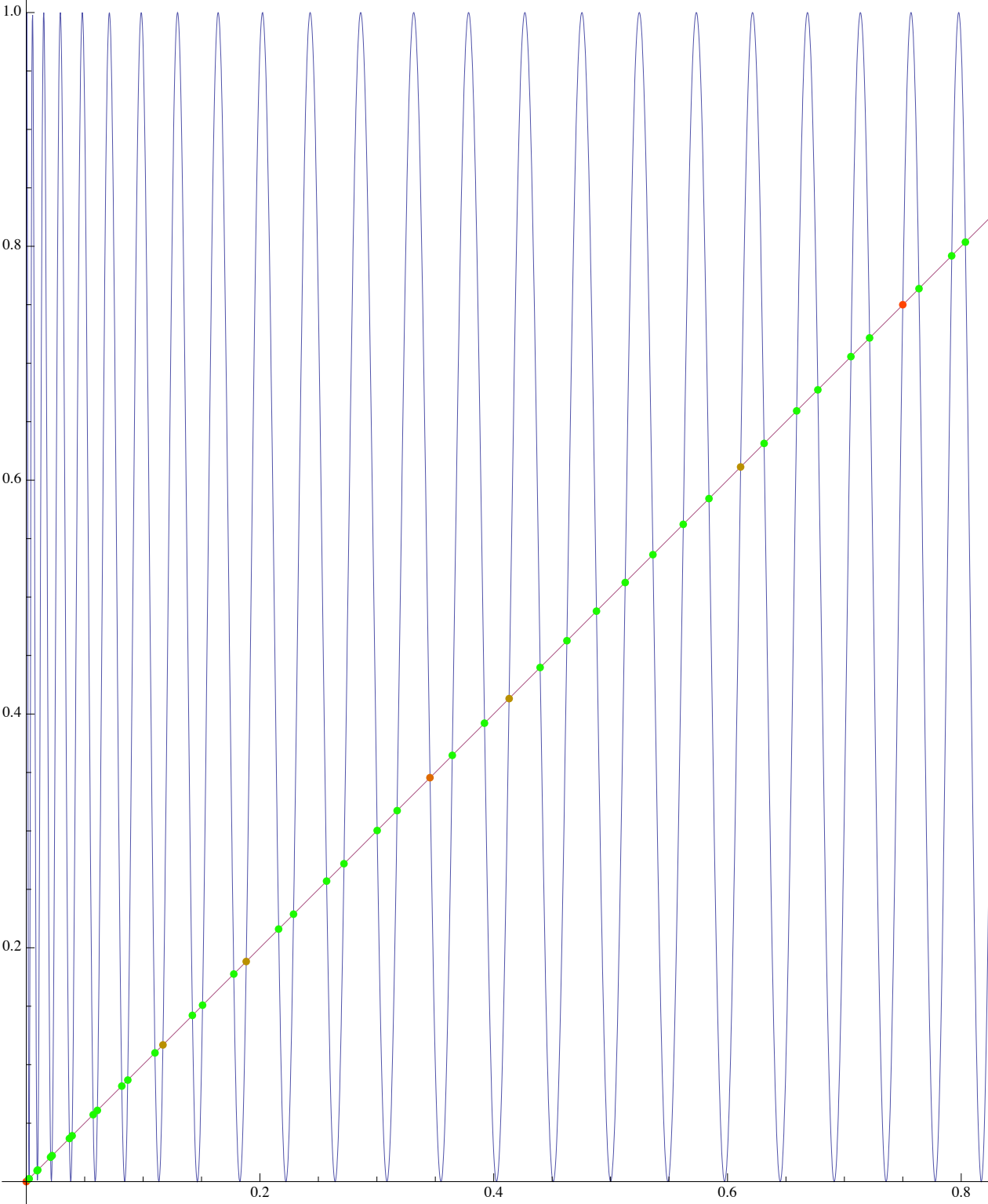
```
{0, 0.75, 0.0793732, 0.292292, 0.571157, 0.82743, 0.979746, 0.00903565,
0.035816, 0.138133, 0.209972, 0.382121, 0.476209, 0.663534, 0.893027, 0.944418,
0.997736, 0.010235, 0.0405211, 0.0896183, 0.155517, 0.235518, 0.326347, 0.424286,
0.525325, 0.625326, 0.720197, 0.806053, 0.879379, 0.937173, 0.97707, 0.997435}
```

`showppf4[5]`

```
GraphicsArray[Partition[cobweb[f4, #, {0, 1.01}, 5] & /@distinctorbits[Drop[p5, 2], 5], 3]]
```



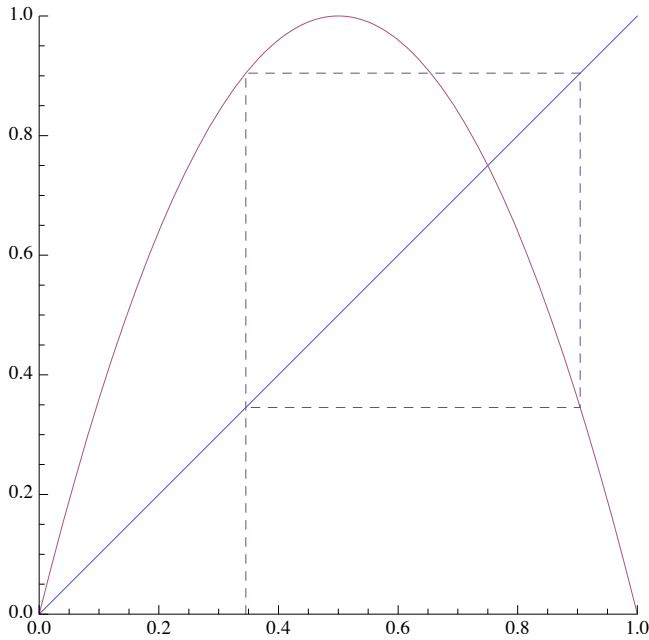
showppf4[6, Medium]



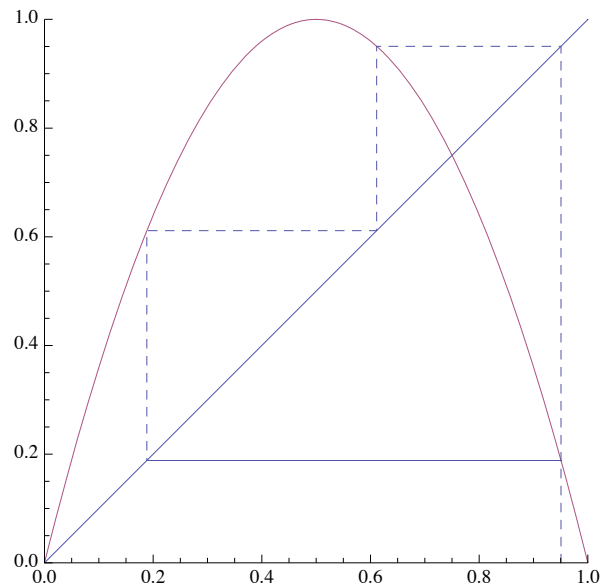
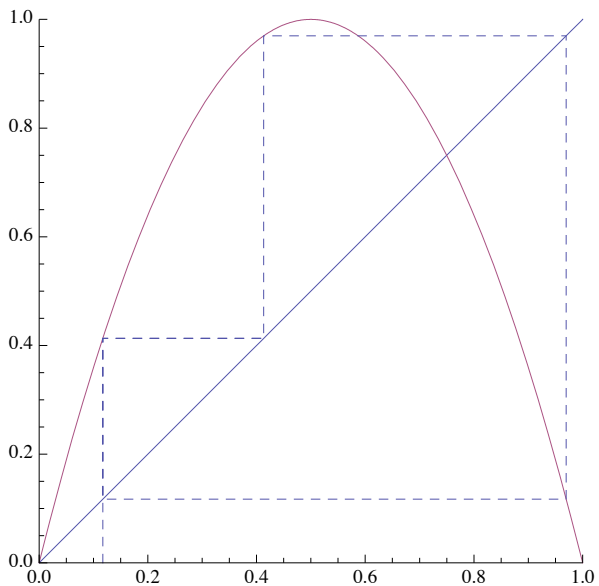
```
p6 = x /. Solve[Nest[f4, x, 6] == x, x] // N // Chop
```

```
{0, 0.75, 0.345492, 0.904508, 0.116978, 0.969846, 0.413176, 0.950484, 0.188255,
0.61126, 0.057272, 0.215968, 0.439732, 0.677302, 0.874255, 0.985471, 0.0222136,
0.0868806, 0.317329, 0.462635, 0.866526, 0.994415, 0.228727, 0.271895, 0.999378,
0.150882, 0.36458, 0.984539, 0.0608892, 0.512465, 0.926645, 0.0392619, 0.562172,
0.898566, 0.00991376, 0.659243, 0.830843, 0.00248461, 0.705644, 0.791872, 0.00233418,
0.00931494, 0.0208771, 0.0369127, 0.0817649, 0.110163, 0.142201, 0.177579,
0.257008, 0.300318, 0.392108, 0.487918, 0.536217, 0.584179, 0.631354, 0.721596,
0.76382, 0.803581, 0.840508, 0.930985, 0.953437, 0.971656, 0.994753, 0.999416}
```

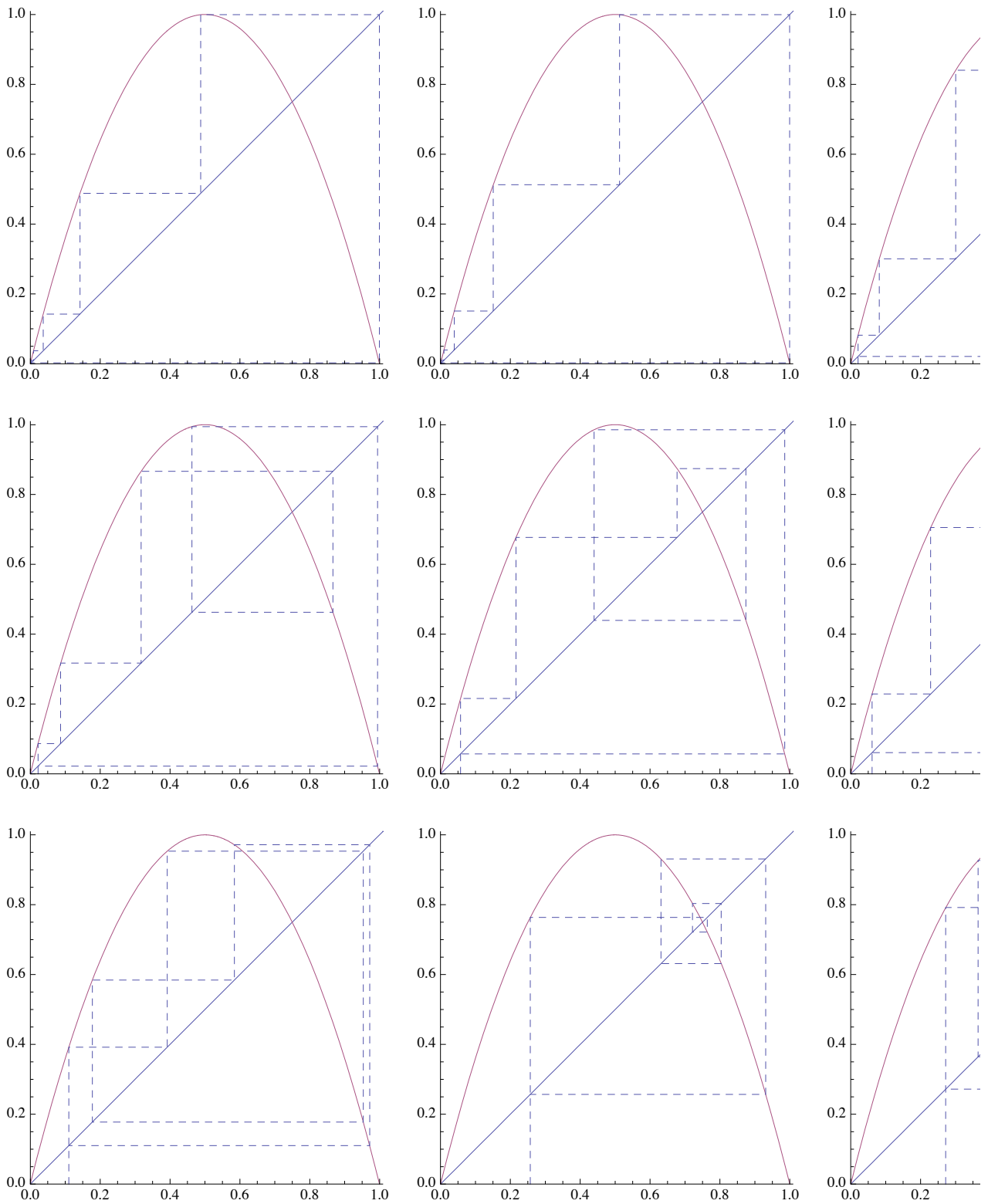
```
cobweb[f4, p6[[3]], {0, 1}, 2]
```



```
GraphicsArray[cobweb[f4, p6[[#]], {0, 1}, 4] & /@ {5, 8}]
```



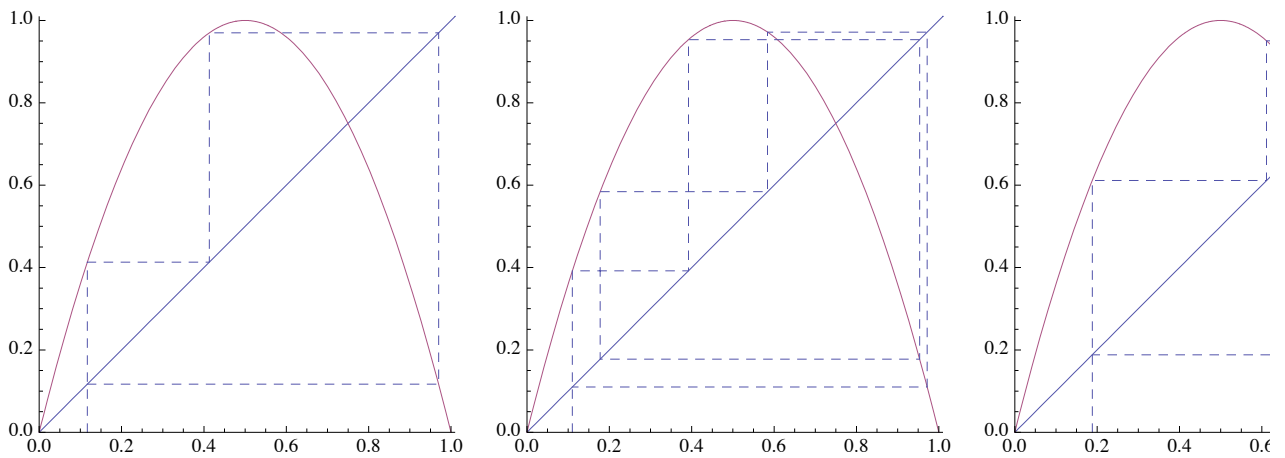
`GraphicsArray[Partition[cobweb[f4, #, {0, 1.01}, 6] & /@ distinctorbits[Drop[p6, 10], 6], 3]]`



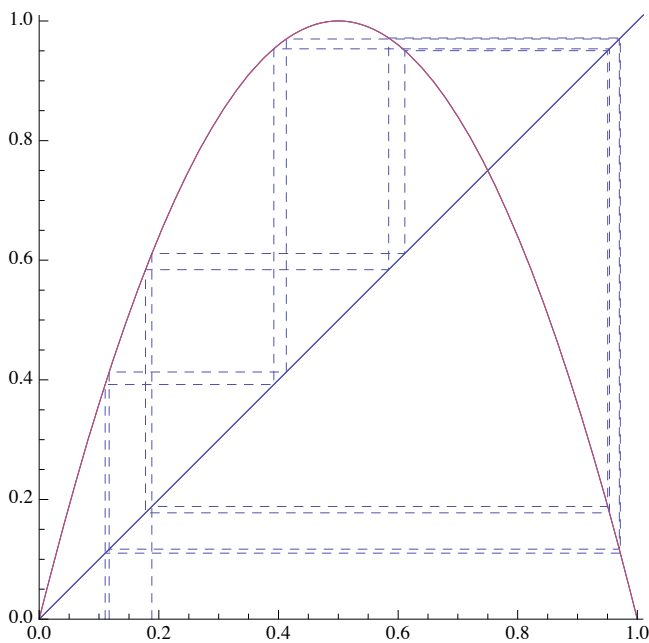
orbit[#, 6] & /@ % // TableForm

0.00233418	0.00931494	0.0369127	0.142201	0.487918	0.999416
0.00248461	0.00991376	0.0392619	0.150882	0.512465	0.999378
0.0208771	0.0817649	0.300318	0.536217	0.840508	0.994753
0.0222136	0.0868806	0.317329	0.462635	0.866526	0.994415
0.057272	0.215968	0.439732	0.677302	0.874255	0.985471
0.0608892	0.228727	0.562172	0.705644	0.830843	0.984539
0.110163	0.177579	0.392108	0.584179	0.953437	0.971656
0.257008	0.631354	0.721596	0.76382	0.803581	0.930985
0.271895	0.36458	0.659243	0.791872	0.898566	0.926645

**GraphicsArray[{cobweb[f4, p6[[5]], {0, 1.01}, 3],
cobweb[f4, %216[[7]], {0, 1.01}, 6], cobweb[f4, p6[[9]], {0, 1.01}, 3]}]**



**Show[{cobweb[f4, p6[[5]], {0, 1.01}, 3],
cobweb[f4, %216[[7]], {0, 1.01}, 6], cobweb[f4, p6[[9]], {0, 1.01}, 3]}]**



Stability of period 6 orbits?

D[Nest[f4, x, 6], x] // Factor

$$\begin{aligned}
 & -4096 (-1 + 2x) (1 - 8x + 8x^2) (1 - 32x + 160x^2 - 256x^3 + 128x^4) \\
 & (1 - 128x + 2688x^2 - 21504x^3 + 84480x^4 - 180224x^5 + 212992x^6 - 131072x^7 + 32768x^8) \\
 & (1 - 512x + 43520x^2 - 1462272x^3 + 25798656x^4 - 275185664x^5 + 1926299648x^6 - \\
 & \quad 9313976320x^7 + 32133218304x^8 - 80648077312x^9 + 148562247680x^{10} - 200655503360x^{11} + \\
 & \quad 196293427200x^{12} - 135291469824x^{13} + 62277025792x^{14} - 17179869184x^{15} + 2147483648x^{16}) \\
 & (1 - 2048x + 698368x^2 - 94978048x^3 + 6885908480x^4 - 308488699904x^5 + 9338794278912x^6 - \\
 & \quad 202785247199232x^7 + 3295260266987520x^8 - 41352285703372800x^9 + \\
 & \quad 410475846508216320x^{10} - 3283806772065730560x^{11} + 21487518225908367360x^{12} - \\
 & \quad 116363175623380697088x^{13} + 526404842105769820160x^{14} - 2003968778223344418816x^{15} + \\
 & \quad 6456334894356662059008x^{16} - 17677237785618240503808x^{17} + 41246888166442561175552x^{18} - \\
 & \quad 82141740303015057817600x^{19} + 139640958515125598289920x^{20} - \\
 & \quad 202406406767568811458560x^{21} + 249477664155375511797760x^{22} - \\
 & \quad 260324519118652707962880x^{23} + 228476306673285621350400x^{24} - \\
 & \quad 167114098595317483044864x^{25} + 100570928113924096131072x^{26} - \\
 & \quad 48915000675954696650752x^{27} + 18740162596631994171392x^{28} - 544178950174431726720x^{29} + \\
 & \quad 1125251388496282648576x^{30} - 147573952589676412928x^{31} + 9223372036854775808x^{32})
 \end{aligned}$$

(% /. x → #) & /@%216

{-64., 64., -64., 64., -64., 64., -64., -63.9705, 63.9436}

NestList[f4, #, 5] & /@%216

```
{
{0.00233418, 0.00931494, 0.0369127, 0.142201, 0.487918, 0.999416},
{0.00248461, 0.00991376, 0.0392619, 0.150882, 0.512465, 0.999378},
{0.0208771, 0.0817649, 0.300318, 0.840508, 0.536217, 0.994753},
{0.0222136, 0.0868806, 0.317329, 0.866526, 0.462635, 0.994415},
{0.057272, 0.215968, 0.677302, 0.874255, 0.439732, 0.985471},
{0.0608892, 0.228727, 0.705644, 0.830843, 0.562172, 0.984539},
{0.110163, 0.392108, 0.953437, 0.177579, 0.584179, 0.971656},
{0.257008, 0.76382, 0.721596, 0.803581, 0.631354, 0.930985},
{0.271895, 0.791872, 0.659243, 0.898566, 0.36458, 0.926645}
}
```

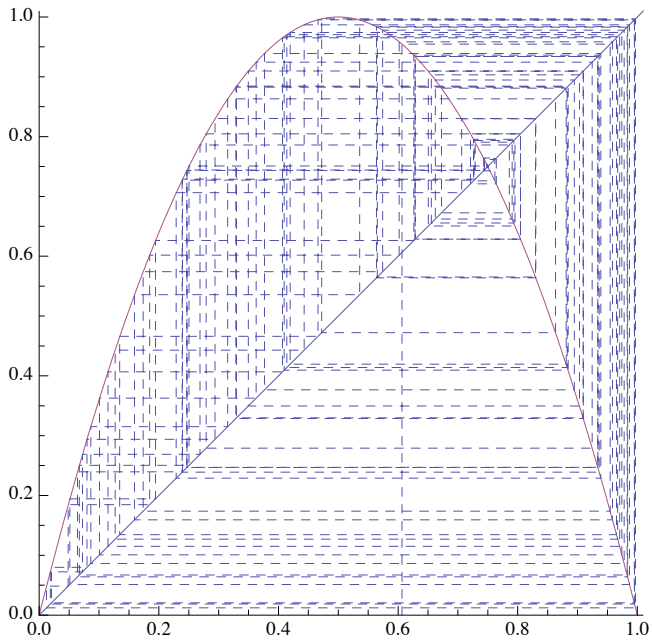
Map[4 (1 - 2 #) &, %, {2}]

```
{
{3.98133, 3.92548, 3.7047, 2.8624, 0.096655, -3.99533},
{3.98012, 3.92069, 3.6859, 2.79295, -0.0997228, -3.99503},
{3.83298, 3.34588, 1.59746, -2.72406, -0.289739, -3.95803},
{3.82229, 3.30496, 1.46136, -2.93221, 0.29892, -3.95532},
{3.54182, 2.27226, -1.41842, -2.99404, 0.482147, -3.88377},
{3.51289, 2.17019, -1.64515, -2.64674, -0.497375, -3.87631},
{3.1187, 0.863137, -3.6275, 2.57937, -0.673428, -3.77325},
{1.94393, -2.11056, -1.77277, -2.42865, -1.05083, -3.44788},
{1.82484, -2.33497, -1.27395, -3.18853, 1.08336, -3.41316}
}
```

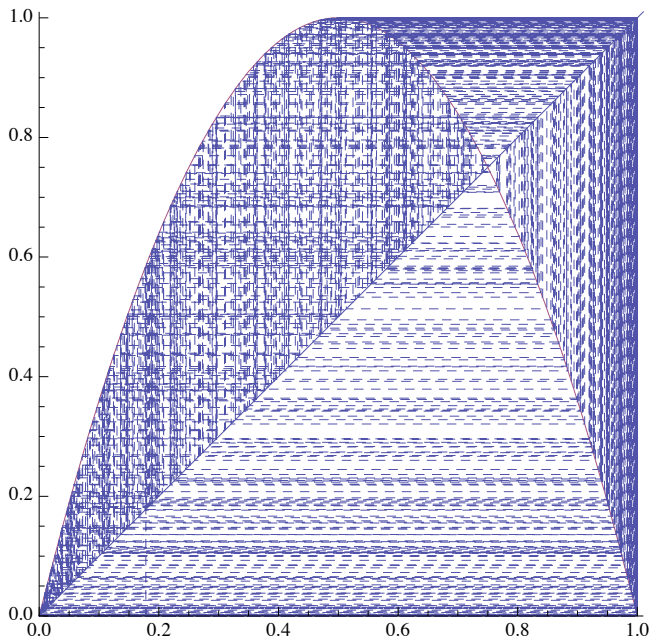
(Times @@ # &) /@%

{-64., 64., -64., 64., -64., 64., -64., -64., 64.}

`cobweb[f4, Random[], {0, 1.01}, 100]`



`cobweb[f4, Random[], {0, 1.01}, 500]`



```
ListPlot[NestList[f4, Random[], 1000], AspectRatio -> 1]
```

