Math 128A, Fall 2011

Recognizing rotations, version 1

(version with graphics and the Cayley transform to follow)

> I shall not today attempt further to define the kinds of material I understand to be embraced within that shorthand description, and perhaps I could never succeed in intelligibly doing so. But I know it when I see it...
>
> *Potter Stewart*

Disclaimer: In what follows, ‘rotation’ means rotation about the origin.

Given real numbers c and s satisfying $c^2 + s^2 = 1$, we defined the rotation $r_{c,s} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$r_{c,\,s}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} c\,x - s\,y \\ s\,x + c\,y \end{pmatrix}.$$

Polar coordinates clarify why $r_{c,\,s}$ should be called a rotation:

$$r_{\cos\phi,\,\sin\phi}\left(\begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \end{pmatrix}\right) = \begin{pmatrix} \cos\phi\,(\rho\cos\theta) - \sin\phi\,(\rho\sin\theta) \\ \sin\phi\,(\rho\cos\theta) + \cos\phi\,(\rho\sin\theta) \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta+\phi) \\ \rho\sin(\theta+\phi) \end{pmatrix}.$$

So, if you believe that a (counter-clockwise) rotation through an angle ϕ about the origin fixes the origin and changes the angle that a nonzero vector $\mathbf{x}$ makes with a fixed line $\mathcal{L}$ through the origin by ϕ, $r_{\cos\phi,\,\sin\phi}$ implements rotation through ϕ.

A rotation of $\mathbb{R}^3$ through an angle ϕ about a specified line $\mathcal{L}$ through the origin transforms every plane perpendicular to $\mathcal{L}$ the way the planar rotation through ϕ transforms $\mathbb{R}^2$. We can easily modify the previous paragraph to algebraically describe rotations about the vertical axis in $\mathbb{R}^3$: define $R_{c,\,s} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$R_{c,\,s}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) := \begin{pmatrix} c\,x - s\,y \\ s\,x + c\,y \\ z \end{pmatrix}.$$

To describe rotation of a vector $\mathbf{x}$ about some other line $\mathcal{L}$ through the origin, we just need to rotate both $\mathcal{L}$ and $\mathbf{x}$ so that $\mathcal{L}$ is rotated onto the vertical axis, rotate as above, then rotate the result back. Specifically, if the rotation $\widetilde{R}$ maps $\mathcal{L}$ onto the vertical axis, then $\widetilde{R}^{-1} \circ R_{\cos\phi,\,\sin\phi} \circ \widetilde{R}$ implements rotation about $\mathcal{L}$ through the angle ϕ. Great, but... we have a chicken–egg situation—how do we explicitly describe the rotation $\widetilde{R}$, and how can we recognize something we can’t describe? To avoid emphasis on a special case, and the associated need to know how to convert the general case to the special one, try an approach that looks the same from every angle.

The inner product determines the angle θ between two nonzero vectors: the equality

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}||\mathbf{y}|}$$

determines θ once a range for arccos has been selected. This definition of angle between vectors is used in all dimensions of Euclidean space $\mathbb{R}^n$, with appropriately adjusted formula for the inner product. We take $\arccos : [-1,1] \to [0,\pi]$; as discussed in lecture, this is convenient in arc length calculations. In the plane we have

$$\left| r_{c,\,s}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\right|^2 = \left|\begin{pmatrix} c\,x - s\,y \\ s\,x + c\,y \end{pmatrix}\right|^2 = (c\,x - s\,y)^2 + (s\,x + c\,y)^2 = \left(c^2 + s^2\right)\left(x^2 + y^2\right) = x^2 + y^2$$

and

$$r_{c,\,s}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = (c\,x - s\,y)x + (s\,x + c\,y)y = c\left(x^2 + y^2\right),$$

so if $x^2 + y^2 \neq 0$, the angle ϕ between $\begin{pmatrix} x \\ y \end{pmatrix}$ and $r_{c,\,s}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$ satisfies

$$\cos\phi = \frac{c\left(x^2 + y^2\right)}{x^2 + y^2} = c,$$

as it should.

So far, so predictable, but we can use this approach to see if an isometry f that fixes the origin is a rotation. Define $\phi_f : \mathbb{R}^2 \backslash \{0\} \to [-1,1]$ by

$$\phi_f(\mathbf{x}) := \frac{f(\mathbf{x}) \cdot \mathbf{x}}{|\mathbf{x}|^2}.$$

Since f is assumed to be an origin-fixing isometry, $|f(\mathbf{x})| = |\mathbf{x}|$, and hence $\phi_f(\mathbf{x})$ gives the cosine of the angle between $\mathbf{x}$ and $f(\mathbf{x})$. It follows that f is a rotation if and only if ϕ_f is constant. If f is a rotation, ϕ_f determines the angle of rotation modulo π.

Informal exercises (not to be turned in).
First exercise: Convince yourself that if f is a reflection of $\mathbb{R}^2$ across a line through the origin, ϕ_f takes all values in $[0,\pi]$: If $\mathbf{x}$ is in the line of reflection, $f(\mathbf{x}) = \mathbf{x}$, and hence $\phi_f(\mathbf{x}) = \arccos 1 = 0$; if $\mathbf{x}$ is perpendicular to the line of reflection, $f(\mathbf{x}) = -\mathbf{x}$, and hence $\phi_f(\mathbf{x}) = \arccos(-1) = \pi$. If $\mathbf{u}$ is a unit vector in the line of reflection, using the explicit expression $\rho_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} - 2\,(\mathbf{x}\cdot\mathbf{u})\mathbf{u}$ for reflection across the line perpendicular to the unit vector $\mathbf{u}$, we have

$$\phi_{\rho_{\mathbf{u}}}(\mathbf{x}) = \frac{(\mathbf{x} - 2\,(\mathbf{x}\cdot\mathbf{u})\mathbf{u})\cdot\mathbf{x}}{|\mathbf{x}|^2} = \frac{|\mathbf{x}|^2 - 2\,(\mathbf{x}\cdot\mathbf{u})^2}{|\mathbf{x}|^2} = \text{[figure it out]}.$$

If $\mathbf{u}$ is a unit vector, $\mathbf{y}$ is a unit vector perpendicular to $\mathbf{u}$, and $\mathbf{x} = \cos\theta\,\mathbf{u} + \sin\theta\,\mathbf{y}$, then $\phi_{\rho_{\mathbf{u}}}(\mathbf{z}) =$ [figure it out].
Second exercise (easier): Convince yourself that if $f(s\,\mathbf{x}) = s\,f(\mathbf{x})$ for some $s \neq 0$ and $\mathbf{x} \neq 0$, then $\phi_f(s\,\mathbf{x}) = \phi_f(\mathbf{x})$. Consequence: if f is linear, then ϕ_f can be regarded as a function of lines through the origin.

What have we gained—or lost—by bringing the inner product into the discussion? We've lost explicit coordinate dependence, which is arguably a gain.

The definition of ϕ_f makes just as much sense for an isometry of $\mathbb{R}^3$ that fixes the origin as it does for an isometry of the plane. The key difference is that we don't expect ϕ_f to be constant on all of $\mathbb{R}^3\backslash\{0\}$, just on some plane through the origin (minus the origin itself). If there is a nonzero vector $\mathbf{u}$ such that $\phi_f(\mathbf{u}) = 1$ and ϕ_f is constant on the plane (with the origin deleted) perpendicular to $\mathbf{u}$, then f is a rotation about $\mathbf{u}$ with the angle of rotation determined modulo π by that constant.

Analogous to the situation in $\mathbb{R}^2$, if f is a reflection across a plane through the origin and $\mathbf{x}$ is in the plane of reflection, $\phi_f(\mathbf{x}) = 1$; if $\mathbf{x}$ is perpendicular to the plane of reflection, $\phi_f(\mathbf{x}) = -1$. One possible rotation/reflection test: a nontrivial rotation determines a 'rotation detector map' sending only a line (minus the origin) to 1, but the rotation detector for a nontrivial reflection maps an entire plane to 1. (Obviously, the convenience and utility of this test depends on the circumstances.)

Two reflections across planes through the origin determine a rotation

We can use the rotation detection method developed above to show that the composition of two reflections across planes through the origin is a rotation.

A reflection across a plane through the origin in $\mathbb{R}^3$ can be implemented by choosing a unit vector $\mathbf{u}$ perpendicular to that plane and defining

$$\rho_{\mathbf{u}}(\mathbf{x}) := \mathbf{x} - 2(\mathbf{u}\cdot\mathbf{x})\mathbf{u}$$

as in $\mathbb{R}^2$. As before, if $\mathbf{x}$ is perpendicular to $\mathbf{u}$ (and hence in the plane of reflection), $\rho_{\mathbf{u}}(\mathbf{x}) = \mathbf{x}$; $\rho_{\mathbf{u}}(s\,\mathbf{u}) = -s\,\mathbf{u}$ for any scalar s.

$$\rho_{\mathbf{u}}(\mathbf{x})\cdot\rho_{\mathbf{u}}(\mathbf{y}) = (\mathbf{x} - 2(\mathbf{u}\cdot\mathbf{x})\mathbf{u})\cdot(\mathbf{y} - 2(\mathbf{u}\cdot\mathbf{y})\mathbf{u}) = \mathbf{x}\cdot\mathbf{y} - 4(\mathbf{u}\cdot\mathbf{x})(\mathbf{u}\cdot\mathbf{y})(1 - \mathbf{u}\cdot\mathbf{u}) = \mathbf{x}\cdot\mathbf{y},$$

so $\rho_{\mathbf{u}}$ is an origin-preserving isometry of $\mathbb{R}^3$ that preserves in the inner product as well as length.

Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent unit vectors, and set $f := \rho_{\mathbf{v}} \circ \rho_{\mathbf{u}}$. (If $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, $\rho_{\mathbf{u}} = \rho_{\mathbf{v}}$ and $\rho_{\mathbf{v}} \circ \rho_{\mathbf{u}}$ is the identity map.) If $\mathbf{x} \neq 0$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, then

$$f(\mathbf{x}) = \rho_{\mathbf{v}}(\rho_{\mathbf{u}}(\mathbf{x})) = \rho_{\mathbf{v}}(\mathbf{x}) = \mathbf{x},$$

i.e. f fixes every vector in the line perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ (which is the line spanned by $\mathbf{u}\times\mathbf{v}$). Given a nonzero vector $\mathbf{x}$, let ψ denote the angle between $\mathbf{u}$ and $\mathbf{x}$, ψ' denote the angle between $\mathbf{v}$ and $\mathbf{x}$, and θ denote the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\begin{aligned}
f(\mathbf{x})\cdot\mathbf{x} &= \rho_{\mathbf{v}}(\rho_{\mathbf{u}}(\mathbf{x}))\cdot\mathbf{x} \\
&= \rho_{\mathbf{v}}(\mathbf{x} - 2\,(\mathbf{x}\cdot\mathbf{u})\mathbf{u})\cdot\mathbf{x} \\
&= (\mathbf{x} - 2\,(\mathbf{x}\cdot\mathbf{u})\mathbf{u} - 2\,(\mathbf{x}\cdot\mathbf{v})\mathbf{v} + 4\,(\mathbf{x}\cdot\mathbf{u})(\mathbf{u}\cdot\mathbf{v})\mathbf{v})\cdot\mathbf{x} \\
&= \mathbf{x}\cdot\mathbf{x} - 2\,(\mathbf{x}\cdot\mathbf{u})^2 - 2\,(\mathbf{x}\cdot\mathbf{v})^2 + 4\,(\mathbf{x}\cdot\mathbf{u})(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{x}) \\
&= |\mathbf{x}|^2\left(1 - 2\,\cos^2\psi - 2\,\cos^2\psi' + 4\,\cos\psi\,\cos\psi'\,\cos\theta\right).
\end{aligned}$$

If $\mathbf{x} \in \operatorname{span}\{\mathbf{u},\mathbf{v}\}$, i.e. $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{v}$ all lie in the same plane, $|\theta| = |\psi - \psi'|$, and hence $\cos\theta = \cos(\psi - \psi')$; thus

$$\phi_f(\mathbf{x}) = \frac{f(\mathbf{x})\cdot\mathbf{x}}{|\mathbf{x}|^2} = 1 - 2\,\cos^2\psi - 2\,\cos^2\psi' + 4\,\cos\psi\,\cos\psi'\,\cos(\psi - \psi') = \cos 2\,(\psi - \psi') = \cos 2\,\theta.$$

Since ϕ_f maps the line spanned by $\mathbf{u}\times\mathbf{v}$ minus the origin to 1 and the plane perpendicular to that line to the constant value $\cos 2\,\theta$, f is a rotation through an angle equal to $2\,\theta$ modulo π about the line spanned by $\mathbf{u}\times\mathbf{v}$.