Practice Problems for the Final Exam   Math 128A   Fall 2011

1. True/False. You *must* justify your answers.

   (a) If an isometry of $\mathbb{R}^2$ cannot be expressed as the composition of less than three reflections, it is a glide reflection.

   (b) The composition of three reflections across lines through the origin in $\mathbb{R}^2$ is a reflection.

   (c) If two lines in the hyperbolic space $\mathbb{H}$ intersect, their images under any Möbius transformation of $\mathbb{H}$ must also intersect.

2. We can define the angle between two lines in spherical geometry as follows: if $\mathcal{L}_1$ and $\mathcal{L}_2$ are lines in S^2 (great circles from a Euclidean perspective), let $\mathcal{P}_1$ and $\mathcal{P}_2$ denote the planes through the origin in $\mathbb{R}^3$ such that $\mathcal{L}_j$ is the intersection of $\mathcal{P}_j$ with S^2, $j = 1, 2$. Let $\widetilde{\mathcal{L}}_j$ denote the Euclidean line in $\mathcal{P}_j$ perpendicular to $\mathcal{P}_1 \cap \mathcal{P}_2$. Then the angle between $\mathcal{L}_1$ and $\mathcal{L}_2$ is defined to be the angle between $\widetilde{\mathcal{L}}_1$ and $\widetilde{\mathcal{L}}_2$.

   (a) Show that for every line $\mathcal{L}$ in S^2 and every point P on $\mathcal{L}$, there is a unique line $\mathcal{L}'$ in S^2 through P such that the angle between $\mathcal{L}$ and $\mathcal{L}'$ is $\frac{\pi}{2}$. The line is called the perpendicular to $\mathcal{L}$ through P.

   (b) Given two perpendicular lines $\mathcal{L}$ and $\mathcal{L}'$ as in (a), show that there is a unique line $\mathcal{L}''$ that is perpendicular to both $\mathcal{L}$ and $\mathcal{L}'$.

   Hint: Visualize the three coordinate planes in $\mathbb{R}^3$.

3. Prove the Cauchy-Schwartz inequality $|\mathbf{x}|\,|\mathbf{y}| \geq |\mathbf{x} \cdot \mathbf{y}|$ in $\mathbb{R}^2$ (the result and proof also holds in $\mathbb{R}^n$, with the 'obvious' Euclidean inner product and distance function on $\mathbb{R}^n$) by filling in the following outline of a proof:

   - Given two vectors $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^2$, define $f : \mathbb{R} \to [0, \infty)$ by $f(s) := |s\,\mathbf{x} - \mathbf{y}|^2$.
   - Show that if $\mathbf{x} = 0$, the inequality is trivially satisfied (both sides equal 0).
   - If $\mathbf{x} \neq 0$, compute $f\left(\dfrac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|^2}\right)$. Since $f(s) \geq 0$ for all $s \in \mathbb{R}$, conclude that the Cauchy-Schwartz inequality holds in this case as well.

4. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) := \frac{z-i}{z+i}$.

   (a) Show that f maps the real axis $\{x : x \in \mathbb{R}\}$ onto the unit circle $\{z : |z| = 1\}$ centered at 0.

   Hint: Use $\left|\dfrac{w}{z}\right| = \dfrac{|w|}{|z|}$ *for any* $w \in \mathbb{C}$ *and nonzero* $z \in \mathbb{C}$ *and keep in mind that* $|x + i\,y|^2 = x^2 + y^2$ *if* $x, y \in \mathbb{R}$.

   (b) Show that f maps $\mathcal{H} = \{x + i\,y : y > 0\}$ onto the interior $\{z : |z| < 1\}$ of the unit circle centered at 0.

   Hint: Don't get buried in algebra. Proceed as directly as possible to a comparison of $(y-1)^2$ *and* $(y+1)^2$ *and use the condition that* $y > 0$.

   (c) Show that f does *not* map $\mathcal{H}$ onto $\mathcal{H}$.

5. Given a unit vector $\mathbf{u} \in \mathbb{R}^2$, define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\mathbf{v}) := \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\,\mathbf{u}$. Show that

   (a) $L^2 = L$, i.e., given any vector $\mathbf{v} \in \mathbb{R}^2$, $L(L(\mathbf{v})) = L(\mathbf{v})$.

   (b) If $\mathbf{v}$ is in the kernel of L, i.e. $L(\mathbf{v}) = 0$, and $\mathbf{w}$ is in the range of L, i.e. $\mathbf{w} = L(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}^3$, then $\mathbf{v} \cdot \mathbf{w} = 0$.

6. Let A, B, C, D be four distinct points in the Euclidean plane such that B and D are on opposite sides of the line AC, $|AB| = |CD|$, and $|AD| = |BC|$. Prove that the line passing through A and B is parallel to the line passing through C and D.

7. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) := (1 - y, 1 + x)$.

   (a) Show that f is a Euclidean isometry.

   (b) Does f have any fixed points? If so, what are they?

   (c) Does f preserve or reverse orientation (handedness)?

   (d) Is f a translation, a reflection, a rotation, or a glide reflection? Justify your answer.

8. (a) Show that if A and B are invertible 2×2 matrices, with associated linear fractional transformations f_A and f_B, then $f_A = f_B$ (in the sense that $f_A(x) = f_B(x)$ for every $x \in \mathbb{R} \cup \{\infty\}$) if and only if A is a rescaling of B, i.e. $A = k\,B$ for some $k \neq 0$.

   (b) An *equivalence relation* is a binary relation $\sim$ on a set S satisfying

   - $a \sim a$ (reflexivity)
   - $a \sim b$ implies $b \sim a$ (symmetry)
   - $a \sim b$ and $b \sim c$ implies $a \sim c$ (transitivity)

   for all a, b and c in S.

   Show that determining the same linear fractional transformation is an equivalence relation on invertible 2×2 real matrices.