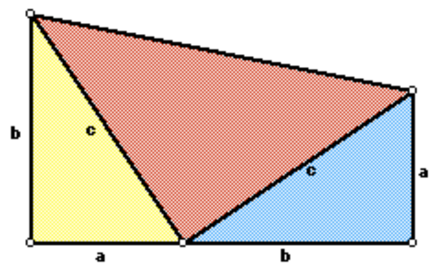
Possible approaches to the practice problems for the midterm
Math 128A, Fall 2011

As we saw at the review, there are many ways of solving these problems. These are just some possibilities.

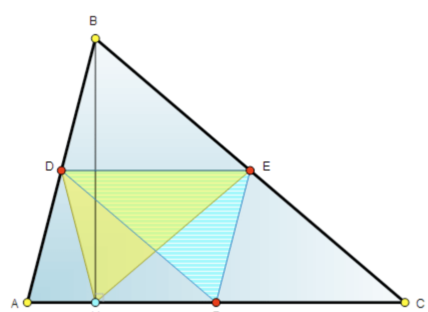
1. The following figure is used in the proof of the Pythagorean Theorem discovered by James Garfield, 20th president of the US (who worked on the proof while a member of Congress, supposedly with assistance from other members):



   Compute the area of this trapezoid two different ways: use this figure, and then use the decomposition of the trapezoid into a rectangle and a triangle to get the standard formula for the area of a trapezoid. Derive the Pythagorean Theorem from the fact that these two approaches must give the same answer. All three triangles are right triangles; assume that $a \geq b$, as indicated in the figure.

   The trapezoid can be decomposed into a rectangle of base $a+b$ and height a and a triangle of base $a+b$ and height $b-a$. (If $a=b$, the triangle is trivial.) Hence the area of the trapezoid is $a(a+b)+\frac{1}{2}(b-a)(a+b)=\frac{1}{2}(a+b)^2$. (This can also be obtained using the formula for the area of a trapezoid.) The combined area of the three triangles in the second figure is $2\times\frac{1}{2}\times a\,b+\frac{1}{2}c^2=\frac{1}{2}\left(c^2+2\,a\,b\right)$. Equating these two expressions for the area yields $c^2=a^2+b^2$.

2. The figure below shows a triangle ABC. If D, E, and F are the midpoints of AB, BC, and AC, respectively, and DEH is the triangle obtained by reflecting BDE across DE, prove that the triangles DEF and DEH are congruent.



   DEH is congruent to BDE, by SSS, since reflection is an isometry. Hence it suffices to show that BDE is congruent to DEF. FEC is similar to ABC, so EF is parallel to

AB. ADF is similar to ABC, so DF is parallel to BC. Hence DBEF is a parallelogram. Since opposite sides of parallelograms have equal length, |BD| = |EF| and |BE| = |DF|. Since BDE and DEF share the common side DE, it follows that BDE and DEF are congruent.

3. Are the following definitions of a rectangle equivalent:

   (a) A quadrilateral with four right angles.

   (b) A parallelogram with at least one right angle.

   If they are, outline a proof of their equivalence; if not, show how they differ.

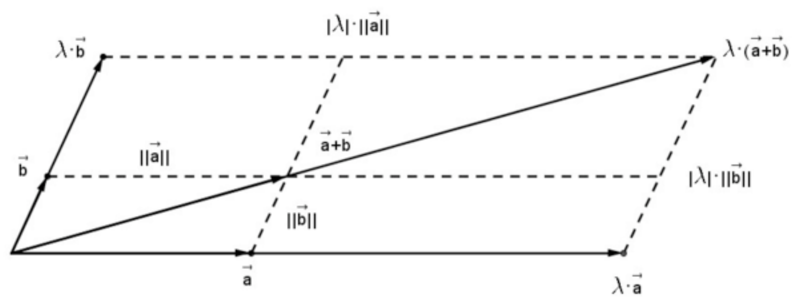
   Yes. First (b). Opposite angles of a parallelogram are equal, call them, say $\theta = \frac{\pi}{2}$ and ϕ. We showed that the interior angles of a triangle sum to π, so the interior angles of a parallelogram which can be decomposed into two triangles, must sum to 2π. Hence $2\phi = 2\pi - 2\theta$, and thus $\phi = \frac{\pi}{2}$ and the parallelogram has four right angles. Now (a). If adjoining sides are perpendicular, then opposite sides must be parallel. Hence the quadrilateral is a parallelogram with four right angles.

4. Explain the relationship between Thales' Theorem and the properties

   $$\lambda(\mathbf{a}+\mathbf{b}) = \lambda\,\mathbf{a} + \lambda\,\mathbf{b} \qquad \text{and} \qquad |\lambda\,\mathbf{a}| = |\lambda|\,|\mathbf{a}| \qquad\qquad \text{for all } \lambda \in \mathbb{R} \text{ and } \mathbf{a},\mathbf{b} \in \mathbb{R}^2$$

   of scalar multiplication in the plane.

   You may use the figure below in your explanation, if you wish. You can be as formal or informal as you like in your identification of points and vectors.



   If we didn't have $\lambda(\mathbf{a}+\mathbf{b}) = \lambda\,\mathbf{a}+\lambda\,\mathbf{b}$, the diagonal line $\lambda(\mathbf{a}+\mathbf{b})$ wouldn't meet the corner of the parallelogram. (If we didn't have $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ for all vectors $\mathbf{x}$ and $\mathbf{y}$, we wouldn't necessarily even have the parallelogram, since adding $\lambda\,\mathbf{b}$ to $\lambda\,\mathbf{a}$ wouldn't necessarily give the same result as adding $\lambda\,\mathbf{a}$ to $\lambda\,\mathbf{b}$, and hence the two 'arms' of the figure might not meet.)

   Clunky point notation to vector notation transition: Given a line segment AB and two parallel lines $\mathcal{L}$ and $\mathcal{M}$ intersecting that segment, with $\mathcal{L}$ passing through A, assume that A lies at the origin (or translate the entire figure so that this is the case). Let C denote the intersection of AB with $\mathcal{M}$. Let $\mathbf{a}$ be contained in $\mathcal{L}$ and let $\mathbf{b}$ be another vector such that $\mathbf{a}+\mathbf{b} = \mathbf{c}$, where $\mathbf{c}$ is the vector from the origin to the point C. Let $\lambda = |AB|/|AC|$, so that the vector from the origin to AB is $\lambda\,\mathbf{c} = \lambda(\mathbf{a}+\mathbf{b})$. (Since λ is a ratio of lengths, *it is non-negative, so $|\lambda| = \lambda$ in this set-up.) If the traditional Thales Theorem is to be true—as it should, since we want $|\ |$ to be a Euclidean distance function—we must have $|\lambda\,\mathbf{b}|/|\mathbf{b}| = |AB|/|AC| = \lambda$, and hence $|\lambda\,\mathbf{b}| = \lambda\,|\mathbf{b}| = |\lambda|\,|\mathbf{b}|$.*

5. (a) Describe how the figure below, including the dashed lines, can be constructed from the triangle ABC using a compass and straightedge. You don't have to explain the details of any constructions we've already done in class or homework.

   We showed how to construct midpoints of line segments, so we can construct the points D, E, and F. Any two points can be connected by a straight line, using the straightedge, so we can construct the line segments AF, BE, and CD. We showed that we can construct a line parallel to any given line and passing through any given point, so we can construct the line passing through C and parallel to AB, and the passing through B and parallel to AC; these lines are not parallel, so they intersect at a unique point, N. Finally, if we construct the line segment AN, it contains the point F, since the diagonals of a parallelogram intersect at their midpoints, and F is the midpoint of the diagonal BC.

   (b) What are the values of the ratios $\dfrac{|GE|}{|BG|}$, $\dfrac{|DG|}{|GC|}$, and $\dfrac{|GF|}{|AG|}$?

   We showed in class that the centroid lies $\frac{2}{3}$ of the way from a vertex to the opposite midpoint, so the ratios all equal $\frac{1}{2}$.

   (c) Where are the centroids of the triangle ABC and the parallelogram $ACNB$?

   We showed in class that the centroid of a triangle is at the intersection of the medians, namely G.

   Now for the parallelogram: Placing one vertex at the origin and labeling the neighboring vertices $\mathbf{u}$ and $\mathbf{v}$, so that the remaining vertex is $\mathbf{u}+\mathbf{v}$, the centroid is the average $\frac{1}{4}(\mathbf{0}+\mathbf{u}+\mathbf{v}+(\mathbf{u}+\mathbf{v})) = \frac{1}{2}(\mathbf{u}+\mathbf{v})$, which is the midpoint of the diagonal connecting $\mathbf{u}$ and $\mathbf{v}$. It was shown in Section 4.3 that the diagonals of a parallelogram bisect each other, so the centroid lies at the intersection of the diagonals, namely F.