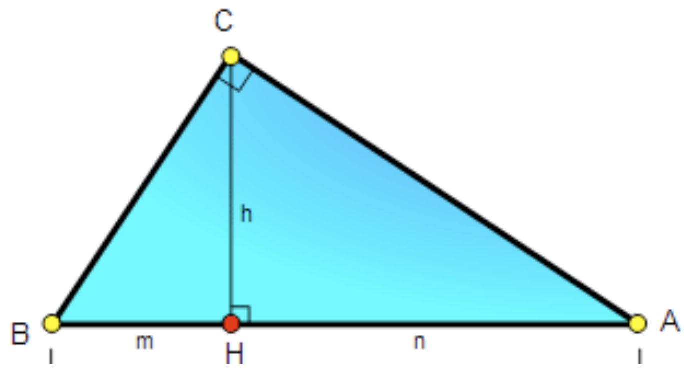
Midterm solution possibilities Math 128A Fall 2011

These are only some possible solutions—-there are others that work just as well, and probably some that are arguably better. The text in small caps is not part of the solution; it indicates comments based on the solutions and attempted solutions seen in the exams, and will be modified as the grading progresses.

1. The figure below shows a right triangle ACB. Prove that $h^2 = m\,n$. Here $h = |CH|$, $m = |BH|$, and $n = |AH|$; CH is perpendicular to AB.



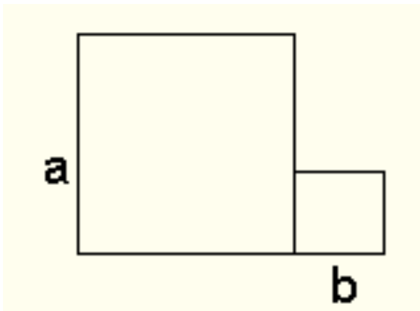
Since ABC is a right triangle, the Pythagorean Theorem implies that

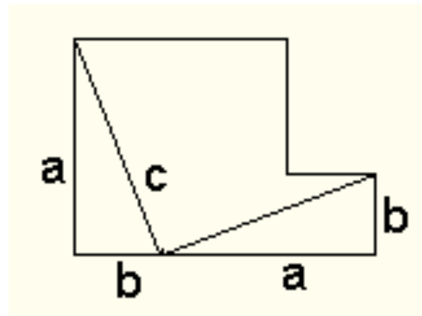
$$|AC|^2 + |BC|^2 = |AB|^2 = (m+n)^2.$$

Similarly, since BCH and ACH are right triangles, $m^2 + h^2 = |BC|^2$ *and* $n^2 + h^2 = |AC|^2$. *Substituting these results into the, first equation yields*

$$(m+n)^2 = \left(n^2 + h^2\right) + \left(m^2 + h^2\right), \qquad \text{and hence} \qquad m\,n = h^2.$$

2. Another proof of the Pythagorean Theorem: Join the two squares shown in the left hand picture into a single region, then divide that region into three pieces as shown in the right hand figure. Construct a square by rearranging the pieces. (Hint: leave at least one point in each of the three pieces in its original position.)





Describe your manipulations of the shapes precisely using the language we've developed in the course and prove the Pythagorean Theorem by relating the areas of the original and new figures.

Rotate the left hand triangle counterclockwise by $\frac{\pi}{2}$ about its upper vertex, so that its side of length a coincides with the upper edge of length a of the peculiar shape. Rotate the right hand triangle clockwise by $\frac{\pi}{2}$ about its upper vertex, so that the sides of length $a$and b coincides with the corresponding sides of the peculiar shape. The result is a square of side length c. Since the original figure is the union of a square with side length a and a square of side length b, it has area a^2+b^2; the new square has area c^2; hence $a^2+b^2=c^2$.

3. Let $ACNB$ be the parallelogram with sides AC, CN, NB, and BA in the figure below.

   (a) Show that $|AF| < \frac{1}{2}(|AB| + |AC|)$.

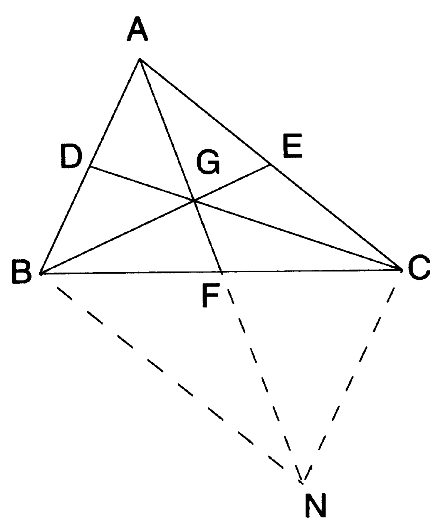
   The triangle inequality implies that $|AN| < |AB| + |BN|$. Since $ACNB$ is a parallelogram, the lengths of the opposite sides are equal; in particular, $|BN| = |AC|$. Hence $|AN| < |AB| + |AC|$. Since the diagonals of a parallelogram intersect at their midpoints, $|AF| = \frac{1}{2}|AN| < \frac{1}{2}(|AB| + |AC|)$.

   (b) Explain why this bound, and analogous bounds on the other two median lengths, implies that the sum of the lengths of the medians of a triangle is less than the length of its perimeter.

   Analogous arguments for the other medians show that $|BE| < \frac{1}{2}(|BA| + |BC|)$ and $|CD| < \frac{1}{2}(|CA| + |CB|)$. Summing these three inequalities gives

$$|AF|+|BE|+|CD| < \frac{1}{2}(|AB|+|AC|+|BA|+|BC|+|CA|+|CB|) = |AB|+|BC|+|CA|,$$

   since $|BA| = |AB|$ etc.



4. The number $\rho = \frac{1}{2}(1+\sqrt{5})$ is called the golden ratio; a rectangle with sides in this ratio is called a golden rectangle. Show that a golden rectangle can be constructed using a compass and straightedge as follows:

- Construct a square ABCD.
- Find the midpoint M of AB.
- Find a point E such that B lies on AE and $|\mathrm{MC}| = |\mathrm{ME}|$.
- Construct the perpendicular to AE passing through E.
- Let F denote the intersection of that perpendicular with the line through C and D.
- AEFD is a golden rectangle.

You must show both that each step can be carried out using only a straightedge and compass, and that AEFD really is a golden rectangle. If some part of the construction was described in lecture, the text, or in one of the assigned homework problems, you can cite that.

- *It was a homework problem to construct a square using a straightedge and compass.*
- *Construction of the midpoint using a straightedge and compass was covered in the text and in lecture.*
- *We can extend the line segment AB arbitrarily far using the straightedge, and use the compass to measure the distance* $|\mathrm{MC}|$ *and find the point* E *satisfying* $|\mathrm{ME}|$*.*
- *The construction of the perpendicular to a given line passing through a given point on that line is given in the text and was shown in lecture.*
- *Since* AB*, and hence* AE*, is parallel to* CD*, the perpendicular constructed in the last step is also perpendicular to* CD*; hence it intersects* CD *at a unique point, which can be found by adequately extending the constructed perpendicular using the straightedge.*
- *Since* $|\mathrm{MB}| = \frac{1}{2}|\mathrm{AB}|$ *and* $|\mathrm{MC}|^2 = |\mathrm{MB}|^2 + |\mathrm{BC}|^2 = (\frac{1}{2}|\mathrm{AB}|)^2 + |\mathrm{AB}|^2 = \frac{5}{4}|\mathrm{AB}|^2$,

$$|\mathrm{AE}| = |\mathrm{AM}| + |\mathrm{ME}| = \frac{1}{2}|\mathrm{AB}| + |\mathrm{MC}| = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)|\mathrm{AB}| = \rho\,|\mathrm{AB}| = \rho\,|\mathrm{AD}|.$$

5. Let $\mathbf{u}$ be a unit length vector, i.e. $|\mathbf{u}| = 1$. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(\mathbf{x}) = \mathbf{x} - 2(\mathbf{u} \cdot \mathbf{x})\mathbf{u}.$$

(a) Is f is a Euclidean isometry? If so, is f a translation, rotation, reflection, or a glide reflection?

$$\begin{aligned} |f(\mathbf{x})|^2 &= |\mathbf{x} - 2(\mathbf{u}\cdot\mathbf{x})\mathbf{u}|^2 \\ &= (\mathbf{x} - 2(\mathbf{u}\cdot\mathbf{x})\mathbf{u}) \cdot (\mathbf{x} - 2(\mathbf{u}\cdot\mathbf{x})\mathbf{u}) \\ &= \mathbf{x}\cdot\mathbf{x} - 2(\mathbf{u}\cdot\mathbf{x})(\mathbf{x}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{x}) + (2(\mathbf{u}\cdot\mathbf{x}))^2\mathbf{u}\cdot\mathbf{u} \\ &= |\mathbf{x}|^2 - 4(\mathbf{u}\cdot\mathbf{x})^2\left(1 - |\mathbf{u}|^2\right) \\ &= |\mathbf{x}|^2 \end{aligned}$$

implies that f is an isometry.

If $\mathbf{x}$ is perpendicular to $\mathbf{u}$, i.e. $\mathbf{u}\cdot\mathbf{x} = 0$, then $f(\mathbf{x}) = \mathbf{x}$, so f fixes all vectors perpendicular to $\mathbf{u}$. Reflections are the only nontrivial isometries fixing an entire line of vectors, so f is a reflection.

(b) Describe the effect of the transformation f in words, without using the dot product or its coordinate formula in your description; you may supplement your description with a sketch if that seems helpful, but don't need to.

f reflects vectors across the line through the origin perpendicular to $\mathbf{u}$.

MANY OF THE ATTEMPTS AT THIS PROBLEM INVOLVE SERIOUS NOTATION CONFUSION. TWO PRESUMABLY FAMILIAR NOTATIONS—$x\,y$ DENOTING THE PRODUCT OF TWO SCALARS x AND y, AND AB DENOTING THE LINE SEGMENT FROM THE POINT A TO THE POINT B—ARE SOMETIMES BEING SCRAMBLED WITH SCALAR MULTIPLICATION $s\,\mathbf{x}$ OR THE INNER PRODUCT $\mathbf{x}\cdot\mathbf{u}$. IN THE PROBLEM, $\mathbf{x}$ AND $\mathbf{u}$ ARE VECTORS ($\mathbf{u}$ IS EXPLICITLY DESCRIBED AS A VECTOR AND $\mathbf{x}$ MUST BE A VECTOR, SINCE f TAKES ELEMENTS OF $\mathbb{R}^2$ AS INPUT), BUT ARE FREQUENTLY BEING CONVERTED TO SCALARS AND BACK. $\mathbf{u}\cdot\mathbf{x}$ IS A SCALAR, SO $2(\mathbf{u}\cdot\mathbf{x})\mathbf{u}$ IS A RESCALING OF $\mathbf{u}$ AND WILL EQUAL $2\,|\mathbf{u}|^2\mathbf{x}$ ONLY IN THE SPECIAL CASE THAT $\mathbf{x}$ AND $\mathbf{u}$ ARE PARALLEL. I WILL COME UP WITH SOME ADDITIONAL VECTOR EXERCISES, SO THAT WE CAN HOPEFULLY HAVE THIS ALL STRAIGHTENED OUT BY THE END OF THE QUARTER.