Math 128A, Fall 2011

Exercises on parametrized lines and linear fractional transformations

Due Wednesday, November 9

1. *Gram-Schmidt orthonormalization in* $\mathbb{R}^2$*:* Given a pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$, with $\mathbf{x} \neq 0$, show that there is a unique scalar λ such that $\mathbf{y} - \lambda\,\mathbf{x}$ is perpendicular to $\mathbf{x}$.

2. Show that given a line $\mathcal{L}$ in $\mathbb{R}^2$, we can find a vector $\mathbf{c}$ and unit length vector $\mathbf{d}$ such that $\mathcal{L} = \mathcal{L}(\mathbf{c}, \mathbf{d})$ and $\mathbf{c}$ is perpendicular to $\mathbf{d}$. Is $\mathbf{c}$ uniquely determined? Is $\mathbf{d}$?

3. Show that given a line $\mathcal{L}$ in $\mathbb{R}^2$, we can find a pair (r, θ), satisfying $r \geq 0$, $0 \leq \theta < 2\,\pi$, and
$$\mathcal{L} = \ell(r, \theta) := \mathcal{L}\left(\begin{pmatrix} -r\,\sin\theta \\ r\,\cos\theta \end{pmatrix}, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right).$$
Is r uniquely determined? Is θ?

4. Compute the linear fractional transformation $\tau(\,\cdot\,; r_1, \theta_1, r_2, \theta_2) : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ determined by projection from $\ell(r_1, \theta_1)$ to $\ell(r_2, \theta_2)$. (The 'eye' is at the origin.) Use trigonometric identities to show that this function depends on θ_1 and θ_2 only through their difference.

5. Given $r_j > 0$ and $0 \leq \theta_j < 2\,\pi$, $j = 1, 2, 3$, compute the composition
$$\tau_{123} := \tau(\,\cdot\,; r_2, \theta_2, r_3, \theta_3) \circ \tau(\,\cdot\,; r_1, \theta_1, r_2, \theta_2),$$
i.e.
$$\tau_{123}(x) = \tau(\tau(x; r_1, \theta_1, r_2, \theta_2); r_2, \theta_2, r_3, \theta_3) \qquad \text{for all } x \in \mathbb{R} \cup \{\infty\}.$$
How is this composition related to $\tau(\cdot; r_1, \theta_1, r_3, \theta_3)$?