Projections of parametrized lines, and linear fractional transformations

Version 2

Given a line $\mathcal{L}$ in $\mathbb{R}^2$, we can find vectors $\mathbf{d} \neq 0$ and $\mathbf{c}$ such that

$$\mathcal{L} = \mathcal{L}(\mathbf{c}, \mathbf{d}) := \{\mathbf{p}(s; \mathbf{c}, \mathbf{d}) : s \in \mathbb{R}\}, \qquad \text{where} \qquad \mathbf{p}(s; \mathbf{c}, \mathbf{d}) := \mathbf{c} + s\,\mathbf{d}.$$

The vectors $\mathbf{c}$ and $\mathbf{d}$ are *not* uniquely determined:

$$\mathbf{p}((s-q)/r; \mathbf{c} + q\,\mathbf{d}, r\,\mathbf{d}) = \mathbf{p}(s; \mathbf{c}, \mathbf{d})$$

for any $q, s \in \mathbb{R}$ and any nonzero $r \in \mathbb{R}$ implies that $\mathcal{L}(\mathbf{c} + q\,\mathbf{d}, r\,\mathbf{d}) = \mathcal{L}(\mathbf{c}, \mathbf{d})$ for any $q \in \mathbb{R}$ and any nonzero $r \in \mathbb{R}$. $\mathcal{L}(\mathbf{c}, \mathbf{d})$ passes through the origin iff $\mathbf{c}$ and $\mathbf{d}$ are linearly dependent.

First consider the following problem: given a vector $\mathbf{v}$ and a parametrized line $\mathcal{L}(\mathbf{c}, \mathbf{d})$ not passing through the origin, find the point of intersection, if any, of the line $\mathcal{L}(\mathbf{c}, \mathbf{d})$ and the line through the origin and $\mathbf{v}$. We seek parameter values s and y for which

$$s\,\mathbf{v} = \mathbf{p}(s; 0, \mathbf{v}) = \mathbf{p}(y; \mathbf{c}, \mathbf{d}) = \mathbf{c} + y\,\mathbf{d}.$$

This is a vector equation in $\mathbb{R}^2$, and hence is equivalent to a pair of scalar equations. We *could* use the standard Euclidean coordinates to split this into two scalar equations, but there's no reason to assume that $\mathcal{L}(0, \mathbf{v})$ and $\mathcal{L}(\mathbf{c}, \mathbf{d})$ are nicely aligned with the x– and y–axes. What matters far more is the extent to which $\mathcal{L}(0, \mathbf{v})$ and $\mathcal{L}(\mathbf{c}, \mathbf{d})$ are aligned with each other, so we'll use a vector–to–scalar conversion that will emphasize that. Let $\Delta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denote the map taking a pair of vectors in $\mathbb{R}^2$ to the determinant of the matrix with columns equal to those vectors. In components,

$$\Delta(\mathbf{x}, \mathbf{y}) = x_1 y_2 - x_2 y_1.$$

We recall (of course, sure... no problem) that the determinant has the following properties:

1. the determinant equals zero iff the columns (or rows) of the matrix are linearly dependent, and hence $\Delta(\mathbf{x}, \mathbf{y}) = 0$ iff $\mathbf{x}$ and $\mathbf{y}$ are parallel;

2. the determinant is a linear function with respect to each column (or row), and hence $\Delta(a\,\mathbf{x} + b\,\mathbf{y}, \mathbf{z}) = a\,\Delta(\mathbf{x}, \mathbf{z}) + b\,\Delta(\mathbf{y}, \mathbf{z})$ and $\Delta(\mathbf{z}, a\,\mathbf{x} + b\,\mathbf{y}) = a\,\Delta(\mathbf{z}, \mathbf{x}) + b\,\Delta(\mathbf{z}, \mathbf{y})$.

3. exchanging two columns (or rows) of the matrix changes the sign of the determinant, leaving the magnitude unchanged, and hence $\Delta(\mathbf{y}, \mathbf{x}) = -\Delta(\mathbf{x}, \mathbf{y})$.

We'll split the vector equation into two scalar ones by applying $\Delta(\mathbf{c}, \cdot)$ and $\Delta(\mathbf{d}, \cdot)$ to both sides of the equation:

$$\Delta(\mathbf{c}, s\,\mathbf{v}) = \Delta(\mathbf{c}, \mathbf{c} + y\,\mathbf{d}) \qquad \text{and} \qquad \Delta(\mathbf{d}, s\,\mathbf{v}) = \Delta(\mathbf{d}, \mathbf{c} + y\,\mathbf{d}).$$

Since $\mathcal{L}(\mathbf{c}, \mathbf{d})$ doesn't pass through the origin, $\mathbf{c}$ and $\mathbf{d}$ are linearly independent, so the vector equation is satisfied iff these two scalar equations are simultaneously satisfied; in addition, linearly independence implies that $\Delta(\mathbf{c}, \mathbf{d}) \neq 0$. Using the properties of Δ listed above, we can simplify the equations:

$$s\,\Delta(\mathbf{c}, \mathbf{v}) = \Delta(\mathbf{c}, s\,\mathbf{v}) = \Delta(\mathbf{c}, \mathbf{c} + y\,\mathbf{d}) = \Delta(\mathbf{c}, \mathbf{c}) + y\,\Delta(\mathbf{c}, \mathbf{d}) = y\,\Delta(\mathbf{c}, \mathbf{d})$$

and

$$s\,\Delta(\mathbf{d}, \mathbf{v}) = \Delta(\mathbf{d}, s\,\mathbf{v}) = \Delta(\mathbf{d}, \mathbf{c} + y\,\mathbf{d}) = \Delta(\mathbf{d}, \mathbf{c}) + y\,\Delta(\mathbf{d}, \mathbf{d}) = \Delta(\mathbf{d}, \mathbf{c}) = -\Delta(\mathbf{c}, \mathbf{d}).$$

The tidied up system of equations

$$s\,\Delta(\mathbf{c}, \mathbf{v}) = y\,\Delta(\mathbf{c}, \mathbf{d}) \qquad \text{and} \qquad s\,\Delta(\mathbf{d}, \mathbf{v}) = -\Delta(\mathbf{c}, \mathbf{d})$$

is easy to analyse.

Note: In practice, you'd just look for something to whack the equation $s\,\mathbf{v} = \mathbf{c} + y\,\mathbf{d}$ with to 'kill' the $\mathbf{d}$ term, and thus the y–dependence, letting you solve for s; then 'kill' the $\mathbf{c}$ term to make it easy to solve for y. If the properties of the determinant were very familiar, you could go straight from the vector equation to the tidied up pair of scalar equations without any fuss.

Case 1: $\mathbf{d}$ and $\mathbf{v}$ are linearly dependent, and hence $\Delta(\mathbf{d}, \mathbf{v}) = 0$; since $\Delta(\mathbf{c}, \mathbf{d}) \neq 0$, the second equation has no solution. The lines $\mathcal{L}(0, \mathbf{v})$ and $\mathcal{L}(\mathbf{c}, \mathbf{d})$ are parallel and distinct, so they never intersect.

Case 2: $\mathbf{d}$ and $\mathbf{v}$ are linearly independent, and hence $\Delta(\mathbf{d}, \mathbf{v}) \neq 0$. The pair of equations has the unique solution

$$s = -\frac{\Delta(\mathbf{c}, \mathbf{d})}{\Delta(\mathbf{d}, \mathbf{v})} \qquad \text{and} \qquad y = \frac{s\,\Delta(\mathbf{c}, \mathbf{v})}{\Delta(\mathbf{c}, \mathbf{d})} = -\frac{\Delta(\mathbf{c}, \mathbf{v})}{\Delta(\mathbf{d}, \mathbf{v})}.$$

The lines $\mathcal{L}(0, \mathbf{v})$ and $\mathcal{L}(\mathbf{c}, \mathbf{d})$ are not parallel and intersect at the point $\mathbf{c} - \dfrac{\Delta(\mathbf{c}, \mathbf{v})}{\Delta(\mathbf{d}, \mathbf{v})}\,\mathbf{d}$.

If we replace $\mathbf{c}$ with $\mathbf{c}' := \mathbf{c} + q\,\mathbf{d}$ and $\mathbf{d}$ with $\mathbf{d}' := r\,\mathbf{d}$ for $q \in \mathbb{R}$ and nonzero $r \in \mathbb{R}$, we still find the same intersection point (if the lines intersect).

Now consider two distinct lines $\mathcal{L}_1$ and $\mathcal{L}_2$, and a point E not on either line. Using an isometric change of coordinates (translate by E) if needed, we can assume that E is the origin. For each point X on $\mathcal{L}_1$, if the line through E and X intersects $\mathcal{L}_2$, the intersection point is called the projection of X onto $\mathcal{L}_2$ (determined by the point E and line $\mathcal{L}_1$); we will denote the map taking points on$\mathcal{L}_1$ onto their projections onto $\mathcal{L}_2$ by $T : \mathcal{L}_1 \to \mathcal{L}_2$.

Given parametrizations $\mathcal{L}_1 = \mathcal{L}(\mathbf{c}_1, \mathbf{d}_1)$ and $\mathcal{L}_2 = \mathcal{L}(\mathbf{c}_2, \mathbf{d}_2)$, our previous calculations show that the map f given by

$$f(x; \mathbf{c}_1, \mathbf{d}_1, \mathbf{c}_2, \mathbf{d}_2) := -\frac{\Delta(\mathbf{c}, \mathbf{p}(x; \mathbf{c}_1, \mathbf{d}_1))}{\Delta(\mathbf{d}, \mathbf{p}(x; \mathbf{c}_1, \mathbf{d}_1))} = -\frac{\Delta(\mathbf{d}_1, \mathbf{c}_2)x + \Delta(\mathbf{c}_1, \mathbf{c}_2)}{\Delta(\mathbf{d}_1, \mathbf{d}_2)x + \Delta(\mathbf{c}_1, \mathbf{d}_2)}$$

satisfies $T(\mathbf{p}(x; \mathbf{c}_1, \mathbf{d}_1)) = \mathbf{p}(f(x); \mathbf{c}_2, \mathbf{d}_2)$.

Note that $\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel iff $\mathbf{d}_1$ and $\mathbf{d}_2$ are parallel, in which case $\Delta(\mathbf{d}_1, \mathbf{d}_2) = 0$ and f is an affine function

$$f(x) = a\,x + b, \qquad \text{where} \qquad a = \frac{\Delta(\mathbf{d}_1, \mathbf{c}_2)}{\Delta(\mathbf{d}_2, \mathbf{c}_1)} \quad \text{and} \quad b = \frac{\Delta(\mathbf{c}_1, \mathbf{c}_2)}{\Delta(\mathbf{d}_2, \mathbf{c}_1)}.$$

Another approach: 'classic' linear algebra

The vector equation $s\,\mathbf{v} = \mathbf{c} + y\,\mathbf{d}$ is affine (linear plus constant) with respect to s and y and hence can be formulated as a traditional Math 21 style linear algebra problem: If we let A denote the matrix with columns $\mathbf{v}$ and $\mathbf{d}$ and $\mathbf{z}$ denote the vector with entries s and $-y$, then

$$A\,\mathbf{z} = s\,(\text{first column of } A) + (-y)\,(\text{second column of } A) = s\,\mathbf{v} - y\,\mathbf{d},$$

so the equation can be expressed as $A\,\mathbf{z} = \mathbf{c}$. Recall that $\operatorname{range} A = \{A\,\mathbf{z} : \mathbf{z} \in \mathbb{R}^2\}$ equals the span of the columns of A. Since $\mathbb{R}^2$ is two dimensional and $\mathbf{d}$ is nonzero, there are only two possibilities in this situation:

- If $\mathbf{v}$ and $\mathbf{d}$ are linearly dependent, then $\operatorname{range} A = \operatorname{span}\{\mathbf{d}\} = \{t\,\mathbf{d} : t \in \mathbb{R}\}$; since $\mathbf{c}$ is assumed not to be a multiple of $\mathbf{d}$, the equation $A\,\mathbf{z} = \mathbf{c}$ has no solution.

- If $\mathbf{v}$ and $\mathbf{d}$ are linearly independent, then $\operatorname{range} A = \mathbb{R}^2$; A is invertible, and hence $A\,\mathbf{z} = \mathbf{c}$ has the unique solution $\mathbf{z} = A^{-1}\mathbf{c}$. In components,

$$A^{-1}\mathbf{c} = \frac{1}{\det A}\begin{pmatrix} d_2 & -d_1 \\ -v_2 & v_1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{v_1 d_2 - v_2 d_1}\begin{pmatrix} c_1 d_2 - c_2 d_1 \\ c_2 v_1 - c_1 v_2 \end{pmatrix} = \frac{1}{\Delta(\mathbf{v},\mathbf{d})}\begin{pmatrix} \Delta(\mathbf{c},\mathbf{d}) \\ \Delta(\mathbf{v},\mathbf{c}) \end{pmatrix}.$$

  Since $y = -z_2$, we have the same answer as before (whew!).

Dividing by zero

What happens if the denominator of $f(x;\mathbf{c}_1,\mathbf{d}_1,\mathbf{c}_2,\mathbf{d}_2)$ equals zero? We get infinity. No way around that—we can put in a finite value of x and get infinity out. On the other hand, if we input infinity, we typically get back a well-defined finite value:

$$\lim_{x\to\infty} f(x;\mathbf{c}_1,\mathbf{d}_1,\mathbf{c}_2,\mathbf{d}_2) = -\lim_{x\to\infty}\frac{\Delta(\mathbf{d}_1,\mathbf{c}_2)\,x + \Delta(\mathbf{c}_1,\mathbf{c}_2)}{\Delta(\mathbf{d}_1,\mathbf{d}_2)\,x + \Delta(\mathbf{c}_1,\mathbf{d}_2)} = -\frac{\Delta(\mathbf{d}_1,\mathbf{c}_2)}{\Delta(\mathbf{d}_1,\mathbf{d}_2)}$$

if $\Delta(\mathbf{d}_1,\mathbf{d}_2) \neq 0$, i.e. if $\mathbf{d}_1$ and $\mathbf{d}_2$ aren't linearly dependent, i.e. $\mathcal{L}_1$ and $\mathcal{L}_2$ aren't parallel. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel, $\lim_{x\to\infty}|f(x)| = \lim_{x\to\infty}|a\,x + b| = \infty$, since $a \neq 0$. So... we declare ∞ to be a valid value and declare the domain and range of f to be $\mathbb{R} \cup \{\infty\}$. (Note: there is no $-\infty$.)

The infinitely distant observer

The text claims that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel and the 'eye' is infinitely far away, the transformation f is simply a translation in the real line: $f(x) = x + t$ for some $t \in \mathbb{R}$. Why is this true? Let's consider what happens as we push both $\mathcal{L}_1$ and $\mathcal{L}_2$ further and further away from the origin. We need to push both lines in the same direction at the same rate, since we could just as well imagine the 'eye' retreating from a pair of fixed lines; hence we shift the vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ by a common vector $\frac{1}{\epsilon}\,\mathbf{r}$ and consider what happens as ϵ goes to zero (and hence $\frac{1}{\epsilon}$ goes to infinity). Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel, we can take $\mathbf{d}_1 = \mathbf{d}_2 = \mathbf{d}$, and our previous calculation shows that $f(x) = a(\epsilon)\,x + b(\epsilon)$, where

$$a(\epsilon) = \frac{\Delta(\mathbf{d},\mathbf{c}_2 + \frac{1}{\epsilon}\,\mathbf{r})}{\Delta(\mathbf{d},\mathbf{c}_1 + \frac{1}{\epsilon}\,\mathbf{r})} \quad\text{and}\quad b(\epsilon) = \frac{\Delta(\mathbf{c}_1 + \frac{1}{\epsilon}\,\mathbf{r},\mathbf{c}_2 + \frac{1}{\epsilon}\,\mathbf{r})}{\Delta(\mathbf{d},\mathbf{c}_1 + \frac{1}{\epsilon}\,\mathbf{r})}.$$

Regrouping terms using the properties of determinants, we have

$$a(\epsilon) = \frac{\Delta(\mathbf{d}, \mathbf{c}_2) + \frac{1}{\epsilon}\,\Delta(\mathbf{d}, \mathbf{r})}{\Delta(\mathbf{d}, \mathbf{c}_1) + \frac{1}{\epsilon}\,\Delta(\mathbf{d}, \mathbf{r})} = \frac{\epsilon\,\Delta(\mathbf{d}, \mathbf{c}_2) + \Delta(\mathbf{d}, \mathbf{r})}{\epsilon\,\Delta(\mathbf{d}, \mathbf{c}_1) + \Delta(\mathbf{d}, \mathbf{r})}$$

and

$$b(\epsilon) = \frac{\Delta(\mathbf{c}_1, \mathbf{c}_2) + \frac{1}{\epsilon}\,\Delta(\mathbf{r}, \mathbf{c}_2) + \frac{1}{\epsilon}\,\Delta(\mathbf{c}_1, \mathbf{r})}{\Delta(\mathbf{d}, \mathbf{c}_1) + \frac{1}{\epsilon}\,\Delta(\mathbf{d}, \mathbf{r})} = \frac{\epsilon\,\Delta(\mathbf{c}_1, \mathbf{c}_2) - \Delta(\mathbf{c}_2, \mathbf{r}) + \Delta(\mathbf{c}_1, \mathbf{r})}{\epsilon\,\Delta(\mathbf{d}, \mathbf{c}_1) + \Delta(\mathbf{d}, \mathbf{r})},$$

so

$$\lim_{\epsilon \to 0} a(\epsilon) = 1 \qquad \text{and} \qquad \lim_{\epsilon \to 0} b(\epsilon) = \frac{\Delta(\mathbf{c}_1 - \mathbf{c}_2, \mathbf{r})}{\Delta(\mathbf{d}, \mathbf{r})}.$$

Note: if $\mathbf{d}_1$ and $\mathbf{d}_2$ are parallel, but not equal, $\lim_{\epsilon \to 0} a(\epsilon) \neq 1$, because we're effectively using different units of measurement on $\mathcal{L}_1$ and $\mathcal{L}_2$.