More on finding the intersection points of two circles

These notes are my take on the approach suggested by the gentleman in the front row, with—*I think*—almost the same 'picture', but possibly different numbers. If I completely misunderstood him, I hope he'll correct me on that. At any rate, this is one way of bringing some visual intuition into play. As we'll see, my first example in class used a *lousy* combination of radii and center points—sorry.

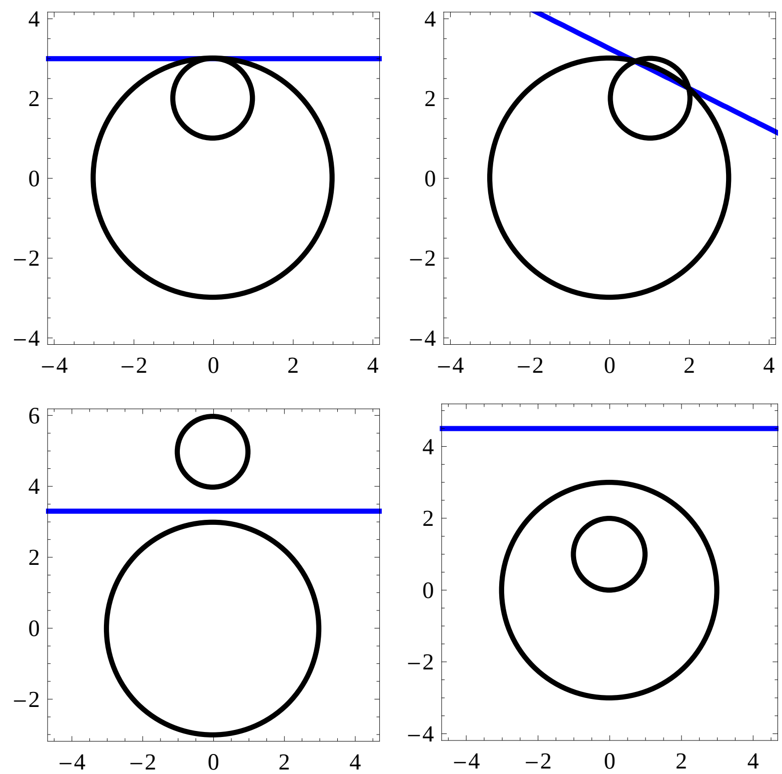

Figure 1: Some sample circle pairs with center points P_j and radii r_j, and the line of points Q satisfying $|QP_1|^2 - r_1^2 = |QP_2|^2 - r_2^2$. The upper row is the pair of examples from class.

Claim: If the centers P_1 and P_2 of the circles are distinct, then the line of points Q satisfying $|QP_1|^2 - r_1^2 = |QP_2|^2 - r_2^2$ is perpendicular to the line $\mathcal{L}$ passing through P_1 and P_2.

"Why should we think this might be true?" rationale #1: The line of points equidistant from the center points P_1 and P_2 is perpendicular to $\mathcal{L}$. No guarantees it works for this problem, too, but it's worth checking into.

"Why should we think this might be true?" rationale #2: (Symmetry addict's perspective.) The figure consisting of the pair of circles doesn't change if reflected across the line $\mathcal{L}$ passing through P_1 and P_2—individual points get mapped to different points, but each circle gets mapped back onto itself. Since reflections preserve distances (we haven't proved that yet, but will soon), the line $\mathcal{M}$ of points Q satisfying $|QP_1|^2 - r_1^2 = |QP_2|^2 - r_2^2$ must also be mapped onto itself by that reflection. Hence $\mathcal{M}$ is either $\mathcal{L}$ itself or a line perpendicular to $\mathcal{L}$. Since we're assuming the circles are distinct, we can rule out $\mathcal{L}$. (Why?) For what follows, we don't even need to rule out $\mathcal{L}$, though.

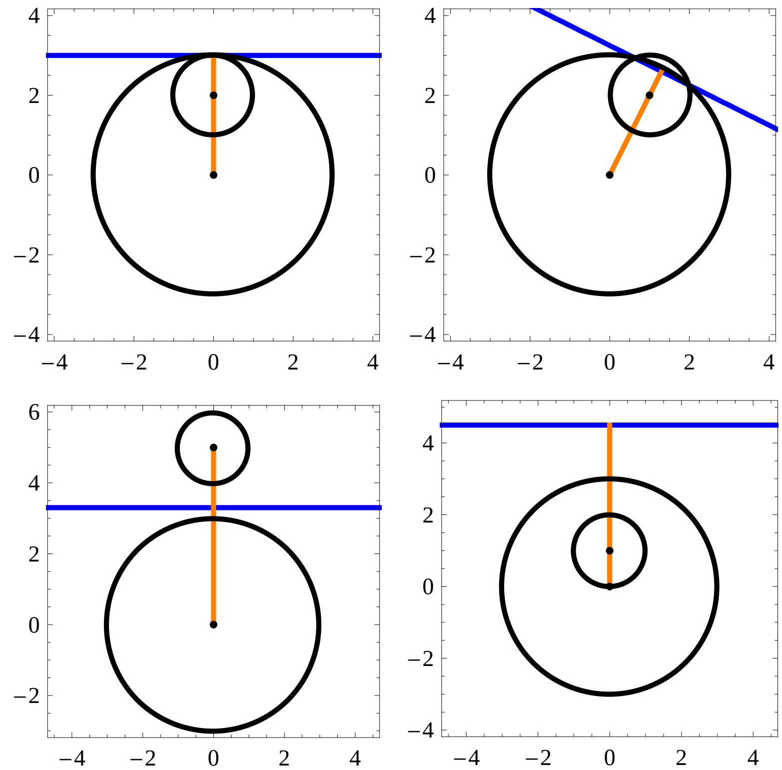

Figure 2: Same as Figure 1, but with segments of the lines $\mathcal{L}$ added in.

Once we've decided $\mathcal{L}$ and lines perpendicular to $\mathcal{L}$ might be relevant, we can choose a coordinate system accordingly. Take coordinates such that one of the axes (I chose the y–axis) is $\mathcal{L}$ and the origin is the center of one of the circles (I chose P_1). In this coordinate system P_1 has coordinates $(0, 0)$ and P_2 has coordinates $(0, d)$ for some real number d, as in the first example I did in class. (This isn't essential, but it tidies up the algebra. All but the upper right hand figure are plotted with respect to this coordinate system.) The equations for the circles are now

$$x^2 + y^2 = r_1^2 \qquad \text{and} \qquad x^2 + (y - d)^2 = r_2^2.$$

Taking the difference of these quadratic equations gives the linear equation $0 = 2\,d\,y + r_2^2 - r_1^2 - d^2$, with unique solution

$$y_* = \frac{d^2 + r_1^2 - r_2^2}{2\,d} = \frac{d}{2} + \left(\frac{r_1 + r_2}{2\,d}\right)(r_1 - r_2).$$

In these coordinates, $\mathcal{M}$ is the horizontal line $y = y_*$. The horizontal line at height $\frac{d}{2}$ is the line of points equidistant from the centers of the circles; $\mathcal{M}$ is offset from that line by a distance that's the product of (the ratio of the average of the radii to the distance between the centers) and (the difference of the radii). Note that in the first example I did in class, $\frac{r_1+r_2}{2\,d} = 1$; that was *not* intentional and I apologize for giving you an atypical example that suggested a simpler formula for y_* than actually works in general.