Additional exercises: cross products and the Cayley transform

Due November 30

1. **The cross product.**

   (a) Show that two vectors $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^3$ are equal if and only if $\mathbf{x} \cdot \mathbf{z} = \mathbf{y} \cdot \mathbf{z}$ for all $\mathbf{z} \in \mathbb{R}^3$. (This is true in all dimensions, but just show it for $\mathbb{R}^3$.)
   Hint: Consider $\mathbf{z} = \mathbf{e}_j$*, the* j*-th Euclidean basis vector.*

   $\mathbf{x} = \mathbf{y} \implies \mathbf{x} \cdot \mathbf{z} = \mathbf{y} \cdot \mathbf{z}$ *is hopefully clear. To show* $\mathbf{x} \cdot \mathbf{z} = \mathbf{y} \cdot \mathbf{z}$ *for all* $\mathbf{z} \implies \mathbf{x} = \mathbf{y}$*, note that* $\mathbf{x} \cdot \mathbf{e}_j = \mathbf{y} \cdot \mathbf{e}_j$ *implies the* j*-th components of* $\mathbf{x}$ *and* $\mathbf{y}$ *are equal; since this is true for* $j = 1, 2, 3$*,* $\mathbf{x} = \mathbf{y}$*.*

   (b) Define the *cross product* of two vectors $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^3$ as follows: for any vector $\mathbf{z} \in \mathbb{R}^3$, $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}$ equals the determinant of the matrix with columns $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. Show that

   i. $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$

   because exchanging two columns in a matrix just changes the sign of the determinant.

   ii. $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = 0 \quad \text{if} \quad \mathbf{z} \in \operatorname{span}\{\mathbf{x}, \mathbf{y}\} = \{a\,\mathbf{x} + b\,\mathbf{y} : a, b \in \mathbb{R}\}$

   because the determinant of a matrix equals zero if the columns are linearly dependent.

   iii. $(s\,\mathbf{x}) \times \mathbf{y} = s(\mathbf{x} \times \mathbf{y})$

   because the determinant is linear 'in each column'

   iv. $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} + \mathbf{y} \cdot (\mathbf{x} \times \mathbf{z}) = 0$

   because the matrix with columns $\mathbf{x}$*,* $\mathbf{z}$*,* $\mathbf{y}$ *(in that order) is obtained from the matrix with columns* $\mathbf{x}$*,* $\mathbf{y}$*,* $\mathbf{z}$ *by exchanging the second and third columns.*

   for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$, $s \in \mathbb{R}$, and invertible matrices A.

   Hint: Use the corresponding properties of the determinant.

   (c) Given $\mathbf{x} \in \mathbb{R}^3$, define the matrix $\widehat{\mathbf{x}}$ by $\widehat{\mathbf{x}}\,\mathbf{y} = \mathbf{x} \times \mathbf{y}$ for all $\mathbf{y} \in \mathbb{R}^3$. Show that if A^T denotes the transpose of the matrix A, so that $\mathbf{x} \cdot A\,\mathbf{y} = (A^T\mathbf{x}) \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, then $\widehat{\mathbf{x}}^T = -\widehat{\mathbf{x}}$, i.e. $\widehat{\mathbf{x}}$ is skew-symmetric (AKA anti-symmetric).
   (b)iv implies that

   $$(\widehat{\mathbf{x}}^T\mathbf{y}) \cdot \mathbf{z} = \mathbf{y} \cdot \widehat{\mathbf{x}}\,\mathbf{z} = \mathbf{y} \cdot (\mathbf{x} \times \mathbf{z}) = -(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = -(\widehat{\mathbf{x}}\,\mathbf{y}) \cdot \mathbf{z}$$

   for any $\mathbf{x}$*,* $\mathbf{y}$*, and* $\mathbf{z}$*.*

2. **The Cayley transform.**

   Define
   $$\operatorname{cay}(\mathbf{x}) := \left(\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^{-1}\left(\mathbb{I} + \tfrac{1}{2}\,\widehat{\mathbf{x}}\right),$$
   where $\mathbb{I}$ denotes the identity matrix. Show the following:

   (a) $(\operatorname{cay}(\mathbf{x}))^T = \operatorname{cay}(-\mathbf{x}) = (\operatorname{cay}(\mathbf{x}))^{-1}$.

   Hint: $\mathbb{I} \mp \frac{1}{2}\,\widehat{\mathbf{x}} = 2\,\mathbb{I} - (\mathbb{I} \pm \frac{1}{2}\,\widehat{\mathbf{x}})$.

Hint: $(A\,\mathbf{x}) \cdot (A\,\mathbf{y}) = \mathbf{x} \cdot A^T A\mathbf{y}$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

We can exchange the order of multiplication:

$$\begin{aligned}\left(\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^{-1}\left(\mathbb{I} + \tfrac{1}{2}\,\widehat{\mathbf{x}}\right) &= \left(\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^{-1}\left(2\,\mathbb{I} - (\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}})\right)\\ &= 2\left(\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^{-1} - \mathbb{I}\\ &= \left(2\,\mathbb{I} - (\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}})\right)\left(\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^{-1}\\ &= \left(\mathbb{I} + \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)\left(\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^{-1}.\end{aligned}$$

(There are lots of other ways of showing this.)

We need some properties of the transpose. You could just quote them from your linear algebra course, but we'll derive them here.

- *Given matrices* A *and* B*, we have*

  $$(A\,B)^T\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A\,B\,\mathbf{y} = (A^T\mathbf{x}) \cdot (B\,\mathbf{y}) = (B^T A^T\mathbf{x}) \cdot \mathbf{y},$$

  for any vectors $\mathbf{x}$ *and* $\mathbf{y}$*, so* $(A\,B)^T = B^T A^T$*. (Here we used the analog of 1(a)—two matrices* C *and* D *are equal if and only if* $\mathbf{x} \cdot C\mathbf{y} = \mathbf{x} \cdot D\mathbf{y}$ *for any vectors* $\mathbf{x}$ *and* $\mathbf{y}$*.)*
- *Transposition is linear:*

  $$\begin{aligned}(s\,A + t\,B)^T\mathbf{x} \cdot \mathbf{y} &= \mathbf{x} \cdot (s\,A\,\mathbf{y} + t\,B\,\mathbf{y})\\ &= s\,(\mathbf{x} \cdot A\,\mathbf{y}) + t\,(\mathbf{x} \cdot B\,\mathbf{y})\\ &= s\,(A^T\mathbf{x} \cdot \mathbf{y}) + t\,(B^T\mathbf{x} \cdot \mathbf{y})\\ &= \left(s\,A^T + t\,B^T\right)\mathbf{x} \cdot \mathbf{y}.\end{aligned}$$
- *Transposition commutes with inversion:*

  $$\left(A^{-1}\right)^T A^T = \left(AA^{-1}\right)^T = \mathbb{I}^T = \mathbb{I}$$

  implies that $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$*, since inverses are unique.*

These properties imply that

$$\begin{aligned}(\mathrm{cay}(\mathbf{x}))^T &= \left(\left(\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^{-1}\left(\mathbb{I} + \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)\right)^T\\ &= \left(\mathbb{I} + \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^T\left(\left(\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^{-1}\right)^T\\ &= \left(\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}}\right))\left(\mathbb{I} + \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^{-1}\\ &= \left(\mathbb{I} + \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^{-1}\left(\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)\\ &= \mathrm{cay}(-\mathbf{x})\end{aligned}$$

and hence, using the second-to-last line of the above string of equalities,

$$(\mathrm{cay}(\mathbf{x}))^T\mathrm{cay}(\mathbf{x}) = \left(\mathbb{I} + \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^{-1}\left(\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)\left(\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^{-1}\left(\mathbb{I} + \tfrac{1}{2}\,\widehat{\mathbf{x}}\right) = \mathbb{I}.$$

(b) Multiplication by a matrix A is an isometry of Euclidean 3-space if $A^T A = \mathbb{I}$.

$$|A\,\mathbf{x}|^2 = (A\,\mathbf{x}) \cdot (A\,\mathbf{x}) = \mathbf{x} \cdot (A^T A\,\mathbf{x}) = \mathbf{x} \cdot \mathbf{x} = |\mathbf{x}|^2$$

for any $\mathbf{x} \in \mathbb{R}^3$*, so multiplication by* A *is an isometry.*

(c) Multiplication by $\mathrm{cay}(\mathbf{x})$ is an isometry.

Combine (a) and (b).

(d) $\det(\mathrm{cay}(\mathbf{x})) = 1$ for any $\mathbf{x} \in \mathbb{R}^3$. You may use without proof the facts that det and cay are continuous maps.

Hint: Use $\mathrm{cay}(\mathbf{0}) = \mathbb{I}$ *and* $\det A = 1$ *or* -1 *if multiplication by* A *is an isometry, and the continuity assertions to show that* $d(s) := \det(\mathrm{cay}(s\,\mathbf{x})) = 1$ *for* $s \in [0,1]$. *(All you need to use about continuity is that* $d(s)$ *can't 'jump' from 1 to* -1.

det A^T = *det* A *and det*$(A\,B) = ($*det* $A)($*det* $B)$ *for any square matrices* A *and* B *imply that*

$$1 = det\,\mathbb{I} = det((\mathrm{cay}(\mathbf{x}))^T \mathrm{cay}(\mathbf{x})) = \left(det\left(\mathrm{cay}(\mathbf{x})^T\right)\right)(det\,\mathrm{cay}(\mathbf{x})) = (det\,\mathrm{cay}(\mathbf{x}))^2,$$

so $det\mathrm{cay}(\,)\mathbf{x} = \pm 1$. *Since the curve* $d(s)$ *described in the hint satisfies* $|d(s)| = 1$ *for any* s,

$$d(0) = \det(\mathrm{cay}(0\,\mathbf{x})) = \det(\mathrm{cay}(0)) = \det \mathbb{I} = 1,$$

and d *is continuous,* $d(s) = 1$ *for any* s.

(e) $\frac{d}{ds}\mathrm{cay}(s\,\mathbf{x})|_{s=0} = \widehat{\mathbf{x}}$.

Hint: Compute the derivative of $\mathbb{I} + \frac{s}{2}\widehat{\mathbf{x}} = \left(\mathbb{I} - \frac{s}{2}\widehat{\mathbf{x}}\right)\mathrm{cay}(s\,\mathbf{x})$ *and use the matrix version of the product rule, namely* $(A\,B)' = A'\,B + A\,B'$, *to figure out the derivative of* $\mathrm{cay}(s\,\mathbf{x})$ *from that.*

$$\begin{aligned}
\tfrac{1}{2}\widehat{\mathbf{x}} &= \frac{d}{ds}\left(\mathbb{I} + \tfrac{s}{2}\widehat{\mathbf{x}}\right)|_{s=0} \\
&= \frac{d}{ds}\left(\mathbb{I} - \tfrac{s}{2}\widehat{\mathbf{x}}\right)\mathrm{cay}(s\,\mathbf{x})|_{s=0} \\
&= \left(-\tfrac{1}{2}\widehat{\mathbf{x}}\,\mathrm{cay}(s\,\mathbf{x}) + \left(\mathbb{I} - \tfrac{s}{2}\widehat{\mathbf{x}}\right)\frac{d}{ds}\mathrm{cay}(s\,\mathbf{x})\right)|_{s=0} \\
&= -\tfrac{1}{2}\widehat{\mathbf{x}} + \frac{d}{ds}\mathrm{cay}(s\,\mathbf{x})|_{s=0}
\end{aligned}$$

implies $\frac{d}{ds}\mathrm{cay}(s\,\mathbf{x})|_{s=0} = \widehat{\mathbf{x}}$.