Additional exercises: cross products and the Cayley transform

Due November 30

1. **The cross product.**

   (a) Show that two vectors $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^3$ are equal if and only if $\mathbf{x} \cdot \mathbf{z} = \mathbf{y} \cdot \mathbf{z}$ for all $\mathbf{z} \in \mathbb{R}^3$. (This is true in all dimensions, but just show it for $\mathbb{R}^3$.)
   Hint: Consider $\mathbf{z} = \mathbf{e}_j$*, the* j*-th Euclidean basis vector.*

   (b) Define the *cross product* of two vectors $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^3$ as follows: for any vector $\mathbf{z} \in \mathbb{R}^3$, $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}$ equals the determinant of the matrix with columns $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. Show that

      i. $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$

      ii. $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = 0 \quad \text{if} \quad \mathbf{z} \in \operatorname{span}\{\mathbf{x}, \mathbf{y}\} = \{a\,\mathbf{x} + b\,\mathbf{y} : a, b \in \mathbb{R}\}$

      iii. $(s\,\mathbf{x}) \times \mathbf{y} = s(\mathbf{x} \times \mathbf{y})$

      iv. $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} + \mathbf{y} \cdot (\mathbf{x} \times \mathbf{z}) = 0$

   for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$, $s \in \mathbb{R}$, and invertible matrices A.

   Hint: Use the corresponding properties of the determinant.

   (c) Given $\mathbf{x} \in \mathbb{R}^3$, define the matrix $\widehat{\mathbf{x}}$ by $\widehat{\mathbf{x}}\,\mathbf{y} = \mathbf{x} \times \mathbf{y}$ for all $\mathbf{y} \in \mathbb{R}^3$. Show that if A^T denotes the transpose of the matrix A, so that $\mathbf{x} \cdot A\,\mathbf{y} = (A^T\mathbf{x}) \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, then $\widehat{\mathbf{x}}^T = -\widehat{\mathbf{x}}$, i.e. $\widehat{\mathbf{x}}$ is skew-symmetric (AKA anti-symmetric).

2. **The Cayley transform.**

   Define
   $$\operatorname{cay}(\mathbf{x}) := \left(\mathbb{I} - \tfrac{1}{2}\,\widehat{\mathbf{x}}\right)^{-1}\left(\mathbb{I} + \tfrac{1}{2}\,\widehat{\mathbf{x}}\right),$$
   where $\mathbb{I}$ denotes the identity matrix. Show the following:

   (a) $(\operatorname{cay}(\mathbf{x}))^T = \operatorname{cay}(-\mathbf{x}) = (\operatorname{cay}(\mathbf{x}))^{-1}$.

   Hint: $\mathbb{I} \mp \frac{1}{2}\,\widehat{\mathbf{x}} = 2\,\mathbb{I} - (\mathbb{I} \pm \frac{1}{2}\,\widehat{\mathbf{x}})$.

   (b) Multiplication by a matrix A is an isometry of Euclidean 3-space if $A^T A = \mathbb{I}$.

   Hint: $(A\,\mathbf{x}) \cdot (A\,\mathbf{y}) = \mathbf{x} \cdot A^T A\mathbf{y}$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

   (c) Multiplication by $\operatorname{cay}(\mathbf{x})$ is an isometry.

   (d) $\det(\operatorname{cay}(\mathbf{x})) = 1$ for any $\mathbf{x} \in \mathbb{R}^3$. You may use without proof the facts that det and cay are continuous maps.

   Hint: Use $\operatorname{cay}(\mathbf{0}) = \mathbb{I}$ *and* $\det A = 1$ *or* -1 *if multiplication by* A *is an isometry, and the continuity assertions to show that* $d(s) := \det(\operatorname{cay}(s\,\mathbf{x})) = 1$ *for* $s \in [0, 1]$*. (All you need to use about continuity is that* $d(s)$ *can't 'jump' from 1 to* -1*.*

   (e) $\frac{d}{ds}\operatorname{cay}(s\,\mathbf{x})|_{s=0} = \widehat{\mathbf{x}}$.

   Hint: Compute the derivative of $\mathbb{I} + \frac{s}{2}\,\widehat{\mathbf{x}} = \left(\mathbb{I} - \frac{s}{2}\,\widehat{\mathbf{x}}\right)\operatorname{cay}(s\,\mathbf{x})$ *and use the matrix version of the product rule, namely* $(A\,B)' = A'\,B + A\,B'$*, to figure out the derivative of* $\operatorname{cay}(s\,\mathbf{x})$ *from that.*