Midterm solution suggestions
Math 106A, Fall 2007

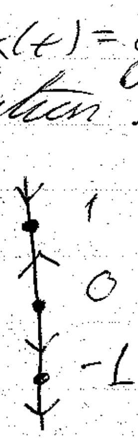
1.a) False. Unless $n=1$, A may consist of multiple smaller blocks, e.g. $A = \mathrm{diag}(\lambda, .., \lambda)$.

b) False. $x(t)=t-1$ and $x(t)=e^{-t}+t-1$ are solutions of *different* IVPs (the first equals -1 at $t=0$, the second equals 0 at $t=0$). The evolution equation can be regrouped as $\dot{x}=f(x,t)$, $f(x,t)=t-x$, which does satisfy the hypotheses of the Uniqueness Thm.

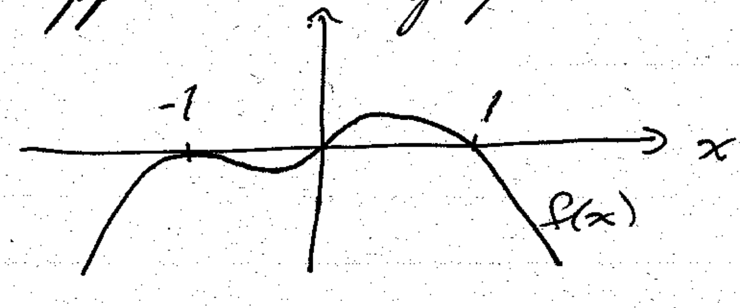
c) True. $x(0)=\frac{e^0+e^{-0}}{2}=1$,
$\dot{x}(0)=\frac{e^0-e^{-0}}{2}=0$, and
$\ddot{x}(t)=\frac{e^t+e^{-t}}{2}=x(t)$.

d) False. If $f(x)$ is a nonlinear function, there is no reason to expect that the sum of two solutions will be a solution. (E.g. $f(x)\equiv 1$, $x(t)=y(t)=t$; $(x+y)(t)=2t$ is *not* a solution.)

2.a) $\dot{x}=f(x)$ has the phase line shown at right:



Hence $f(x)$ should be positive on the interval $(0,1)$, 0 at $1, 0, -1$, and negative everywhere else.
One possibility is $f(x) = x(x+1)^2(1-x)$, with approximate graph



Since $f(x)$ is positive for $0<x<1$, the solution starting at $\frac{1}{2}$ approaches 1 as $t \to \infty$ but never quite reaches it (uniqueness of solutions!), and approaches 0 as $t \to -\infty$.
Since $f(x)$ is negative on $(-1,0)$, the solution starting at $-1/2$ approaches -1 as $t \to \infty$ and 0 as $t \to -\infty$.

b) $f(x,t) = (x^2-1)e^{xt}$ is continuous, with continuous partial derivative
$$\frac{\partial f}{\partial x}(x,t) = \left(2x + (x^2-1)t\right)e^{xt}.$$
Hence the Uniqueness Theorem applies to $\dot{x} = f(x,t)$, which has equilibria $x = \pm 1$.
$|x(t)| < 1 \iff -1 < x(t) < 1$,

so the statement that $|x(t)| < 1$ if $|x(0)| < 1$ is equivalent to the statement that $x(t)$ remains trapped between the two equilibria if it starts between them, which follows from the Uniqueness Thm.

3.a) The first order system
$$\dot{x} = xy \qquad \dot{y} = y + 1$$
is partially decoupled (x depends on y, but y doesn't depend on x), so we solve the y eqn. first.
$$\frac{dy}{y+1} = dt \Rightarrow \ln|y+1| = t + c_1$$
for some constant c_1,
$$\Rightarrow |y+1| = e^{t+c_1}$$
$$\Rightarrow y+1 = k_1 e^{t} \quad \text{for some constant } k_1 \; (= \pm e^{c_1}).$$
$$\Rightarrow y(t) = k_1 e^{t} - 1.$$
Substitute this into the x eqn:
$$\dot{x} = x\,y(t) = x\,(k_1 e^{t} - 1).$$
$$\Rightarrow \frac{dx}{x} = k_1 e^{t} - 1$$
$$\Rightarrow \ln|x| = k_1 e^{t} - t + c_2$$
for some constant c_2
$$\Rightarrow x(t) = k_2 e^{k_1 e^{t} - t} \quad \text{for some constant } k_2.$$

b) $\dot{x} = -x + 2y \quad \dot{y} = -2x - y$

$\Leftrightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Hence the solution of the IVP is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \exp\left(t\begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}\right)\begin{pmatrix} -2 \\ 4 \end{pmatrix}$$
$$= e^{-t}\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix}\begin{pmatrix} -2 \\ 4 \end{pmatrix}$$
$$= e^{-t}\begin{pmatrix} -2\cos(2t) + 4\sin(2t) \\ 2\sin(2t) + 4\cos(2t) \end{pmatrix}.$$

(You can also solve this using the methods of Chpt. 3).

4) For both (a) and (b), use the fact that the eigenvalues of an upper triangular matrix are the diagonal entries.

a) A has the three distinct eigenvalues $1, 2, 3$; hence its JNF is diagonal. $B = \operatorname{diag}(1,2,3)$ (or any permutation of those diagonal entries).

b) This is a special case of the homework problem: "Show that the matrix $A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$

has JNF $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \iff ab \neq 0.$

Here $a = b = 1$.

5) $A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue 2.

$A\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, so $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of A with eigenvalue 0.

$A\begin{pmatrix} 5 \\ -3 \\ 9 \end{pmatrix} = \begin{pmatrix} -10 \\ 6 \\ -18 \end{pmatrix}$, so $\begin{pmatrix} 5 \\ -3 \\ 9 \end{pmatrix}$ is an eigenvector of A with eigenvalue -2.

Hence $B = \mathrm{diag}(2, 0, -2)$ and

$U = \begin{pmatrix} 1 & 1 & 5 \\ 1 & 1 & -3 \\ 1 & -1 & 9 \end{pmatrix}$ satisfy $B = U^{-1}AU$.

$\exp(tB) = \mathrm{diag}(e^{2t}, 1, e^{-2t})$.

6.a) $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so A has eigenvalues 2 and 0; hence a line of equilibria; expansion along positive diagonal, no change along "negative" diagonal.

$\Rightarrow$ upper right phase portrait + lower left image

b) A has a repeated eigenvalue 2, with a 2×2 Jordan block (A is in JNF). The eigenspace is the horizontal axis, which implies the correct phase portrait is upper left. Shearing + expansion $\Rightarrow$ lower right is the correct image.

c) A has eigenvalues $2 \pm 3i$, with flow consisting of expansion + rotation. $\Rightarrow$ lower right phase portrait and upper left image.

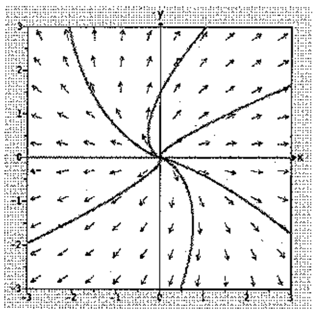
d) $A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$, $A\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$ so A has eigenvalues ± 1, i.e. a saddle. $\Rightarrow$ lower left is the correct ~~ph~~ phase portrait. This A has the same eigenspaces as A, but eigenvalues shifted left (hence smaller images in positive time), $\Rightarrow$ upper right image.

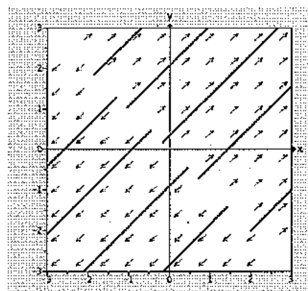
6. Each of the phase portraits below, and the images of the unit square under the flow map at time $t = \frac{1}{4}$ and $t = \frac{1}{2}$, is determined by the linear homogeneous ODE $\dot{x} = Ax$ for one of the matrices a–d; assign each picture the label of the corresponding matrix.

(a) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

(c) $A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ (d) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

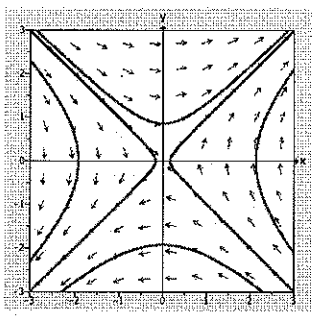
b

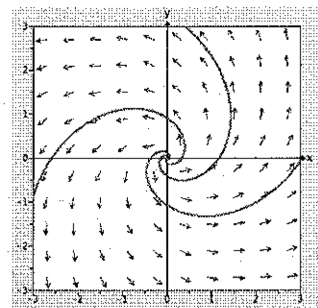




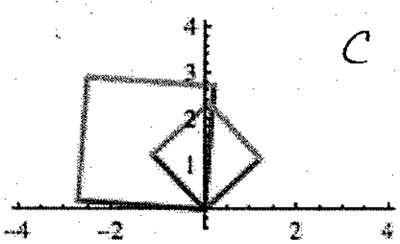
a

d

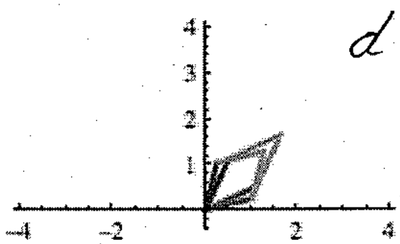




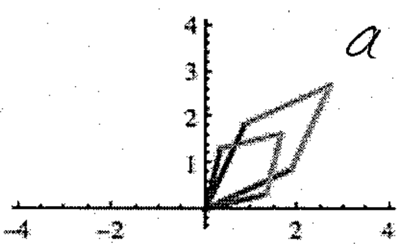
c



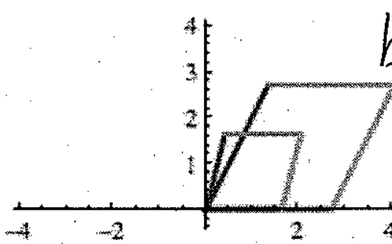
c



d



a



b