Sample midterm-style questions involving the Jordan normal form and the matrix exponential

1) True/false (justify your answer):
If $\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = A \begin{pmatrix} x \\ v \end{pmatrix}$ is the first order system corresponding to the ODE
$\ddot{x} + p\dot{x} + qx = 0$
and the second order ODE has general solution $x(t) = k_1 e^{-3t} + k_2 e^{-2t}$, then $\exp(tA) = \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$.

2) Multiple choice (justify your answer):
If $A \in \mathbb{R}^{3\times 3}$ has eigenvalues 1, 2, 3, then – up to a permutation of the diagonal entries – the Jordan normal form of A is

a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ d) could be either a) or c)

3) Multiple choice (justify your answer):
If A has Jordan normal form

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

then the dimension of the eigenspace of 2 is

a) 1
b) 2
c) 3
d) can't be determined from the given info.

4) True/false (justify your answer):
If $A \in \mathbb{R}^{2\times 2}$ has eigenvalues $\alpha \pm i\beta$, $\beta \neq 0$, then there is an invertible matrix $U \in \mathbb{R}^{2\times 2}$ such that

$$\exp(tA) = U \begin{pmatrix} e^{(\alpha+i\beta)t} & 0 \\ 0 & e^{(\alpha-i\beta)t} \end{pmatrix} U^{-1}.$$

5) If A has eigenvalues $2, -1$, $A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$, and $A\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$
find matrices B and U, U invertible, such that

$$\exp(tA) = U\exp(tB)U^{-1}.$$

Compute $\exp(tB)$.

6) If a) the only eigenvalue of A is 12, with eigenspace $\text{span}\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$,

b) $A\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}13\\14\\1\end{pmatrix}$, and

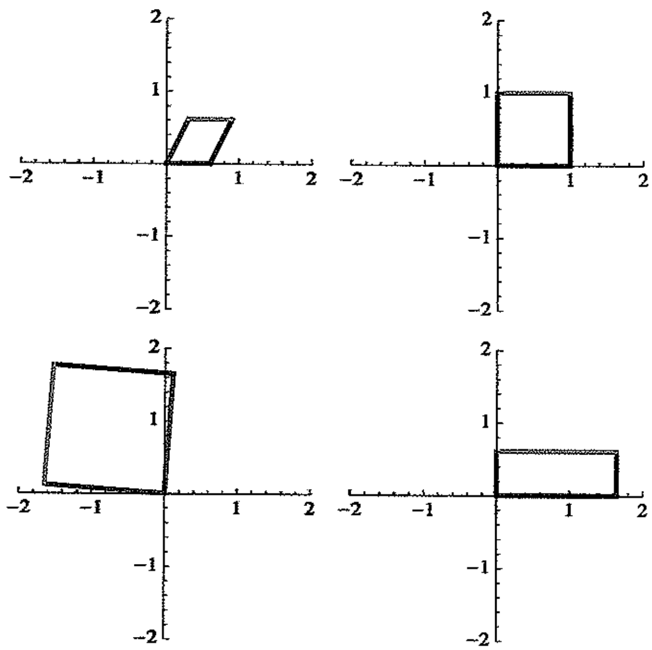
c) $s(A,12)\begin{pmatrix}3\\0\\5\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$

(a-c all hold), find matrices B and C, B in Jordan normal form and C invertible, such that $\exp(tA) = U\exp(tB)U^{-1}$. Compute $\exp(tB)$.

7) If $A \in \mathbb{R}^{2\times 2}$ has eigenvalues $2 \pm 3i$, find a time-dependent family of matrices $B(t)$ such that $\exp(tA) = U B(t) U^{-1}$ for some invertible matrix U. (You don't need to, and can't, say what U is.)

8) Match the following images of the unit square with vertices $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, under $\exp(tA)$, $t = \cancel{1}\,1/2$, with the corresponding matrix A

Out[365]=



a) $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$

c) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

d) $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$.