Some sample Jordan normal form and matrix exponential calculations

■ Definitions

In[212]:= `s[a_, λ_] := a - λ IdentityMatrix[Length[a]]`

LinearSolve, Solve and MatrixExp are built-in *Mathematica* functions. Here are short specs of their syntax:

In[379]:= `? LinearSolve`

LinearSolve[*m*, *b*] finds an *x* which solves the matrix equation *m*.*x* == *b*.
LinearSolve[*m*] generates a LinearSolveFunction[…] which can be applied repeatedly to different *b*. ≫

In[380]:= `? Solve`

Solve[*eqns*, *vars*] attempts to solve an equation or set of equations for the variables *vars*.
Solve[*eqns*, *vars*, *elims*] attempts to solve the equations for *vars*, eliminating the variables *elims*. ≫

In[381]:= `? MatrixExp`

MatrixExp[*m*] gives the matrix exponential of *m*.
MatrixExp[*m*, *v*] gives the matrix exponential of *m* applied to the vector *v*. ≫

■ 4 x 4 example

Note that in *Mathematica* matrices are entered as a list of rows (#@$!).

In[219]:= `a4a = {{-2, 1, 3, -1}, {3, 0, -2, 2}, {1, 1, 2, 1}, {1, -1, -3, 0}};`

In[221]:= `Eigensystem[a4a]`

Out[221]= `{{-1, -1, 1, 1}, {{-5, 3, -1, 5}, {0, 0, 0, 0}, {-1, 1, -1, 1}, {0, 0, 0, 0}}}`

Each eigenvector has only a one dimensional eigenspace. We need to find a preimage under s(A, λ) of each basis vector:

In[130]:= `LinearSolve[s[a4a, -1], {-5, 3, -1, 5}]`

Out[130]= `{2, -3, 0, 0}`

In[131]:= `LinearSolve[s[a4a, 1], {-1, 1, -1, 1}]`

Out[131]= `{0, -1, 0, 0}`

Now build the matrix U, following each eigenvector by its preimage. Since we want our vectors to be the columns, not the rows, of U, we need to transpose the matrix:

In[132]:= `u4a = Transpose[{{-5, 3, -1, 5}, {2, -3, 0, 0}, {-1, 1, -1, 1}, {0, -1, 0, 0}}];`

Verify that we get the correct Jordan normal form. (If we didn't make any mistakes, we knew a priori that we would get this.)

In[234]:=
```
Inverse[u4a].a4a.u4a // MatrixForm
```

Out[234]//MatrixForm=

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In[235]:=
```
MatrixExp[t %] // MatrixForm
```

Out[235]//MatrixForm=

$$\begin{pmatrix} e^{-t} & e^{-t}\, t & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{t} & e^{t}\, t \\ 0 & 0 & 0 & e^{t} \end{pmatrix}$$

In[236]:=
```
u4a.%.Inverse[u4a] // Factor // MatrixForm
```

Out[236]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} e^{-t} \left(2 - 5\, t + 3\, e^{2\, t}\, t\right) & e^{t}\, t & \frac{1}{4} e^{-t} \left(-5 + 5\, e^{2\, t} + 2\, e^{2\, t}\, t\right) & \frac{1}{4} e^{-t} \left(-1 + e^{2\, t} - 10\, t + 4\, e^{2\, t}\, t\right) \\ -\frac{3}{2} e^{-t} \left(-1 + e^{t}\right) \left(1 + e^{t}\right) (-1 + t) & -e^{t} (-1 + t) & -\frac{1}{4} e^{-t} \left(-3 + 3\, e^{2\, t} + 2\, e^{2\, t}\, t\right) & -\frac{1}{4} e^{-t} \left(3 - 3\, e^{2\, t} - 6\, t + 4\, e^{2\, t}\, t\right) \\ \frac{1}{2} e^{-t} \left(-1 + 3\, e^{2\, t}\right) t & e^{t}\, t & \frac{1}{4} e^{-t} \left(-1 + 5\, e^{2\, t} + 2\, e^{2\, t}\, t\right) & \frac{1}{4} e^{-t} \left(-1 + e^{2\, t} - 2\, t + 4\, e^{2\, t}\, t\right) \\ -\frac{1}{2} e^{-t} \left(-5 + 3\, e^{2\, t}\right) t & -e^{t}\, t & -\frac{1}{4} e^{-t} \left(-5 + 5\, e^{2\, t} + 2\, e^{2\, t}\, t\right) & -\frac{1}{4} e^{-t} \left(-5 + e^{2\, t} - 10\, t + 4\, e^{2\, t}\, t \right. \end{pmatrix}$$

In[237]:=
```
% - MatrixExp[t a4a] // Factor // MatrixForm
```

Out[237]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

■ Another 4 x 4 example

In[216]:=
```
a4b = {{-96, 325, 962, -1174}, {-1, 6, 10, -12}, {6, -20, -55, 71}, {13, -43, -126, 157}};
```

In[165]:=
```
Eigensystem[a4b]
```

Out[165]=
```
{{3, 3, 3, 3}, {{-27, -1, 0, 2}, {13, 1, 1, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}
```

The sole eigenvalue of A is 3, with eigenspace span{(-27, -1, 0, 2), (13, 1, 1, 0)}.

In[367]:=
```
s[a4b, 3]
```

Out[367]=
```
{{-99, 325, 962, -1174}, {-1, 3, 10, -12}, {6, -20, -58, 71}, {13, -43, -126, 154}}
```

In[372]:=
```
LinearSolve[s[a4b, 3], {-27, -1, 0, 2}]
```

LinearSolve::nosol : Linear equation encountered that has no solution. ≫

Out[372]=
```
LinearSolve[{{-99, 325, 962, -1174}, {-1, 3, 10, -12},
  {6, -20, -58, 71}, {13, -43, -126, 154}}, {-27, -1, 0, 2}]
```

In[382]:=
```
LinearSolve[s[a4b, 3], {13, 1, 1, 0}]
```

LinearSolve::nosol : Linear equation encountered that has no solution. ≫

Out[382]=
```
LinearSolve[{{-99, 325, 962, -1174},
  {-1, 3, 10, -12}, {6, -20, -58, 71}, {13, -43, -126, 154}}, {13, 1, 1, 0}]
```

We need to get hold of a vector in the eigenspace that' s also in the range of s(A, 3), and hence has a preimage:

In[217]:= **Solve[s[a4b, 3].{w1, w2, w3, w4} == v1 {-27, -1, 0, 2} + v2 {13, 1, 1, 0}, {w1, w2, w3, w4, v1, v2}]**

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[217]= $\left\{\left\{w1\to -\frac{3\,v2}{2}+13\,w3-\frac{27\,w4}{2},\ w2\to -\frac{v2}{2}+w3-\frac{w4}{2},\ v1\to v2\right\}\right\}$

Note that the condition v1 = v2 implies that only a one dimensional subspace,
span{(-27, -1, 0, 2) + (13, 1, 1, 0)} = span{(-14, 0, 1, 2)},
of the eigenspace intersects range(s(A, 3)), i.e. has a preimage under s (A, 3).

In[218]:= **{w1, w2, w3, w4} /. %[[1]]**

Out[218]= $\left\{-\frac{3\,v2}{2}+13\,w3-\frac{27\,w4}{2},\ -\frac{v2}{2}+w3-\frac{w4}{2},\ w3,\ w4\right\}$

We need to do one more "bounce", to find a preimage of this vector:

In[190]:= **Solve[s[a4b, 3].{x1, x2, x3, x4} == %, {x1, x2, x3, x4, w3, w4}]**

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[190]= $\left\{\left\{x1\to -\frac{3\,w4}{4}+13\,x3-\frac{27\,x4}{2}+\frac{43\,y2}{4},\ x2\to -\frac{w4}{4}+x3-\frac{x4}{2}+\frac{13\,y2}{4},\ w3\to \frac{w4}{2}-\frac{y2}{2}\right\}\right\}$

In[191]:= **{%%, {x1, x2, x3, x4}} /. %[[1]]**

Out[191]= $\left\{\left\{-\frac{27\,w4}{2}+13\left(\frac{w4}{2}-\frac{y2}{2}\right)-\frac{3\,y2}{2},\ -y2,\ \frac{w4}{2}-\frac{y2}{2},\ w4\right\},\right.$
$\left.\left\{-\frac{3\,w4}{4}+13\,x3-\frac{27\,x4}{2}+\frac{43\,y2}{4},\ -\frac{w4}{4}+x3-\frac{x4}{2}+\frac{13\,y2}{4},\ x3,\ x4\right\}\right\}$

We can now choose values for the free parameters y2, w4, x3, and x4; y2 needs to be nonzero, or our original element of the eigenspace will be the zero vector!

In[192]:= **% /. {y2 → 1, w4 → 0, x3 → 0, x4 → 0}**

Out[192]= $\left\{\left\{-8,\ -1,\ -\frac{1}{2},\ 0\right\},\ \left\{\frac{43}{4},\ \frac{13}{4},\ 0,\ 0\right\}\right\}$

The corresponding matrix U is :

In[193]:= $\mathbf{u4b = Transpose\left[\left\{\{13, 1, 1, 0\}, \{-27, -1, 0, 2\} + \{13, 1, 1, 0\}, \left\{-8, -1, -\frac{1}{2}, 0\right\}, \left\{\frac{43}{4}, \frac{13}{4}, 0, 0\right\}\right\}\right];}$

In[204]:= **Inverse[u4b].a4b.u4b // MatrixForm**

Out[204]//MatrixForm=

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

In[205]:= **s[%, 3] // MatrixForm**

Out[205]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[206]:= **IdentityMatrix[4] + t % + $\frac{t^2}{2}$ %.% // MatrixForm**

Out[206]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In[207]:= **E^{3t} u4b.%.Inverse[u4b] // Factor // MatrixForm**

Out[207]//MatrixForm=

$$\begin{pmatrix} -e^{3t}\left(-1+99t+7t^2\right) & e^{3t}\,t\,(325+21t) & 2e^{3t}\,t\,(481+35t) & -2e^{3t}\,t\,(587+42t) \\ -e^{3t}\,t & e^{3t}\,(1+3t) & 10e^{3t}\,t & -12e^{3t}\,t \\ \frac{1}{2}e^{3t}\,t\,(12+t) & -\frac{1}{2}e^{3t}\,t\,(40+3t) & -e^{3t}\left(-1+58t+5t^2\right) & e^{3t}\,t\,(71+6t) \\ e^{3t}\,t\,(13+t) & -e^{3t}\,t\,(43+3t) & -2e^{3t}\,t\,(63+5t) & e^{3t}\left(1+154t+12t^2\right) \end{pmatrix}$$

In[208]:= **% - MatrixExp[t a4b] // MatrixForm**

Out[208]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$