Two attempts to match up Chapter 3 with the matrix exponential and Jordan normal form

Extracting exp($t\,A$) from the general solution

If the general solution $x(t) = k_1\, s_1(t) + \ldots + k_n\, s_n(t)$ to the ODE $\frac{d}{dt} x = A\, x$ is already known (say, by using the methods of Chapter 3), exp($t\,A$) can be determined as follows:
Let $S(t) = (s_1(t) \ldots s_n(t))$, the matrix-valued function with columns given by the vector-valued functions s_j. Then $x(t) = S(t)\, k$, where k is the vector with entries $k_1, \ldots, k_n$.
The solution of the IVP $\frac{d}{dt} x = A\, x$ and $x(0) = x_0$ satisfies $x_0 = x(0) = S(0)\, k$; hence $k = S(0)^{-1} x_0$ and

$$x(t) = S(t)\, S(0)^{-1}\, x_0.$$

(If we've really got the general solution, $S(0)$ must be invertible.)
We know that $x(t) = \exp(t\, A)\, x_0$. Since the uniqueness theorem applies to linear homogeneous constant coefficient systems, it follows that $\exp(t\,A) = S(t)\, S(0)^{-1}$.

The single eigenvalue case

If the n × n matrix A has a single eigenvalue λ, the solution of the IVP $\frac{d}{dt} x = A\; x,\; x(0) = x_0$ is

$$x(t) = e^{t\lambda} \sum_{j=0}^{n-1} \frac{t^j}{j!}\, s(A, \lambda)^j\, x_0.$$

Sketch of proof : We know that we can construct an invertible matrix U such that $U^{-1} A\, U$ is in Jordan normal form; since A has only the eigenvalue λ, $U^{-1} A\, U$ has diagonal entries all equal to λ, entries on the first super-diagonal (i.e. the entries $a_{i(i+1)},\; i = 1, \ldots, n-1$), equal to either 0 or 1 (depending on the size of the Jordan blocks), and all other entries equal to 0. (For example, if A diagonalizes, all off-diagonal entries of $U^{-1} A\, U$ are 0; if $U^{-1} A\, U$ is a single n × n Jordan block, all entries on the first super-diagonal are 1.) Hence $s(U^{-1} A\, U, \lambda)$ has 0 entries everywhere except (possibly) on the first super-diagonal; it follows that the n-th power of $s(U^{-1} A\, U, \lambda)$ is the zero matrix (same argument as for N_n, but possibly with more 0' s).

Now use the information from the previous paragraph to compute exp($t\,A$):

$$\begin{aligned}
\exp(t\,A) &= U \exp(t\, U^{-1} A\, U) U^{-1} \\
&= U \left(e^{t\lambda} \exp(t\, s(U^{-1} A\, U,\, \lambda))\right) U^{-1} \\
&= U \left(e^{t\lambda} \sum_{j=0}^{n-1} \frac{t^j}{j!}\, s\left(U^{-1} A\, U,\, \lambda\right)^j\right) U^{-1} \\
&= e^{t\lambda} \sum_{j=0}^{n-1} \frac{t^j}{j!} \left(U\, s\left(U^{-1} A\, U\right),\, \lambda\right)\, U^{-1}\right)^j \\
&= e^{t\lambda} \sum_{j=0}^{n-1} \frac{t^j}{j!}\, s\,(A\,,\, \lambda)^j
\end{aligned}$$

since $U\, B^j\, U^{-1} = U\, B \left(U\, U^{-1}\right) \ldots B\, U^{-1} = \left(U\, B\, U^{-1}\right) \ldots \left(U\, B\;\, U^{-1}\right) = (U\, B\, U^{-1})^j$ for any B and
$U\, s(U^{-1} A\, U,\, \lambda) U^{-1} = U\, (U^{-1} A\, U - \lambda\, \mathrm{id}) U^{-1} = U\, U^{-1} A\, U\, U^{-1} - \lambda\, U\, \mathrm{id}\, U^{-1} = A - \lambda\, \mathrm{id} = s(A\,,\, \lambda)$.

This strategy generalizes the one given in the text in the case of a repeated eigenvalue (for a 2 × 2 matrix) with a one dimensional eigenspace. The efficiency of this approach relative to the one using the complete Jordan normal form depends on the size of the Jordan blocks relative to that of the total matrix. If you have a lot of little Jordan blocks, it's generally efficient to use them; for small n it's no big deal, but if n is large, computing n-1 powers of $s(A\,,\, \lambda)$ is CPU-intensive.