Phase flow and matrix exponential notes

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Consider an autonomous ODE $\dot{x} = f(x)$ on $\mathbb{R}^n$. Assume that solutions exist for all time and all initial conditions. (Not essential, but simplifies the statements.) The *phase flow*, *flow map*, or simply *flow* of f is the family of maps $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$\varphi_0(x) = x \qquad \text{and} \qquad \tfrac{d}{dt}\varphi_t(x) = f(\varphi_t(x))$$

for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Equivalently, for any $x_0 \in \mathbb{R}^n$, $x(t) = \varphi_t(x_0)$ is the solution of the initial value problem $\dot{x} = f(x)$ and $x(0) = x_0$. Sometimes the map $\Phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ given by $\Phi(x, t) := \varphi_t(x)$ is also called the phase flow.

The flow typically cannot be explicitly computed; properties of the flow map are usually implicitly determined by corresponding properties of the vector field f. The extremely important special case in which the flow can be explicitly computed is that of a linear vector field $f(x) = A\,x$ for some matrix $A \in \mathbb{R}^{n\times n}$. In this case, $\varphi_t(x) = \exp(t\,A)x$, where $\exp : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is the matrix exponential

$$\exp(B) := \sum_{j=0}^{\infty} \frac{1}{j!} B^j.$$

This can be shown as follows: Since, by convention, $B^0 = \mathbb{I}$ for matrices (generalizing $x^0 = 1$ for scalars), $\exp(\mathbb{O}) = \mathbb{I}$, and hence $\varphi_0 = \text{identity}$; differentiating $\exp(t\,A)$ with respect to t yields

$$\begin{aligned}
\tfrac{d}{dt} \exp(t\,A) &= \tfrac{d}{dt} \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j \\
&= \sum_{j=1}^{\infty} \frac{t^{j-1}}{(j-1)!} A^j \\
&= A \sum_{j=1}^{\infty} \frac{t^{j-1}}{(j-1)!} A^{j-1} \\
&= A \exp(t\,A).
\end{aligned}$$

Hence $\frac{d}{dt}(\exp(t\,A)x) = A\exp(t\,A)x$ for all $x \in \mathbb{R}^n$, and the flow map for $f(x) = A\,x$ is multiplication by $(\exp(t\,A)$.

A sophisticated but simple way of characterizing the flow map is as the solution of the initial value problem $\varphi_0 = \text{identity}$ and $\frac{d}{dt}\varphi_t = f \circ \varphi_t$ on mappings. Analogously, $\exp(t\,A)$ is the solution of the initial value problem

$$E(0) = \mathbb{I} \qquad \text{and} \qquad \dot{E}(t) = A\,E(t)$$

on $\mathbb{R}^{n\times n}$. The power series for exp and the IVP definition of exp generalize the corresponding power series and IVP for the familiar scalar exponential.

Example 1: $\exp(t\,\mathrm{diag}\,(\lambda_1,\ldots,\lambda_n)) = \mathrm{diag}\left(e^{\lambda_1 t},\ldots,e^{\lambda_n t}\right)$.

Verification: $\mathrm{diag}\left(e^{\lambda_1 0},\ldots,e^{\lambda_n 0}\right) = \mathrm{diag}\,(1,\ldots,1) = \mathbb{I}$ and

$\frac{d}{dt}\mathrm{diag}\left(e^{\lambda_1 t},\ldots,e^{\lambda_n t}\right) = \mathrm{diag}\left(\lambda_1 e^{\lambda_1 t},\ldots,\lambda_n e^{\lambda_n t}\right) = \mathrm{diag}\,(\lambda_1,\ldots,\lambda_n)\,\mathrm{diag}\left(e^{\lambda_1 t},\ldots,e^{\lambda_n t}\right).$

Example 2: $\exp\left(t\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos(\omega\,t) & -\sin(\omega\,t) \\ \sin(\omega\,t) & \cos(\omega\,t) \end{pmatrix}$.

Verification: $\begin{pmatrix} \cos(\omega\,0) & -\sin(\omega\,0) \\ \sin(\omega\,0) & \cos(\omega\,0) \end{pmatrix} = \mathbb{I}$ and

$$\begin{aligned}
&\frac{d}{dt}\begin{pmatrix} \cos(\omega\,t) & -\sin(\omega\,t) \\ \sin(\omega\,t) & \cos(\omega\,t) \end{pmatrix} \\
&= \begin{pmatrix} -\omega\sin(\omega\,t) & -\omega\cos(\omega\,t) \\ \omega\cos(\omega\,t) & -\omega\sin(\omega\,t) \end{pmatrix} \\
&= \left(\cos(\omega\,t)\begin{pmatrix} 0 \\ \omega \end{pmatrix} + \sin(\omega\,t)\begin{pmatrix} -\omega \\ 0 \end{pmatrix} \quad -\sin(\omega\,t)\begin{pmatrix} 0 \\ \omega \end{pmatrix} + \cos(\omega\,t)\begin{pmatrix} -\omega \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} \cos(\omega\,t) & -\sin(\omega\,t) \\ \sin(\omega\,t) & \cos(\omega\,t) \end{pmatrix}.
\end{aligned}$$

If matrices B and C commute, i.e. $B\,C = C\,B$, then $\exp(B+C) = (\exp B)(\exp C)$. (If B and C *don't* commute, this relationship typically doesn't hold. Note that real scalars always commute; hence $e^{x+y} = e^x e^y$ is true for the scalar exponential.) A very important special case is $B = \lambda\,\mathbb{I}$ for some scalar λ; since a multiple of the identity matrix commutes with any other $n\times n$ matrix,

$$\exp(\lambda\,\mathbb{I} + C) = \exp(\lambda\,\mathbb{I})(\exp C) = (e^{\lambda}\,\mathbb{I})(\exp C) = e^{\lambda}\,\exp C.$$

The *Jordan normal form* can be used to explicitly compute the matrix exponential of an arbitrary square matrix. (See Jordan normal form notes.)