Practice Problems   Math 106A   Fall 2007

1. (a) Find the general solution of $4\,y'' + 4\,y' + y = 3\,x\,e^x$.
   (b) Find the general solution of $y'' + y = \sin x + x\,\cos x$.
   (c) Find a particular solution of $y''' - y'' - y' - 2\,y = 4\,x^4$.

2. Consider $H(x, y) = f(x^2) + g(y^2)$. Describe any qualitative features of the solutions phase portrait for the Hamiltonian system with Hamiltonian H that must exist for any choice of differentiable f and g. Sketch the phase portrait for some specific choice of f and g, non-constant and not covered in the text.

   Same thing for the gradient system determined by H. (Use the same specific choice of H for your sketch.)

3. Sketch the phase portraits for $m\,\ddot{x} + k\,x - \epsilon\,\dot{x} = \beta\,x^3$ for both $\epsilon = 0$ and $\epsilon > 0$, and the following values of m, k, and β:

   (a) $m = k = 2$, $\beta = -4$
   (b) $m = 1$, $k = 4$, $\beta = 1$.

   (You should have a total of four phase portraits.)

4. For each of the following matrices, find a matrix B in Jordan normal form and an invertible matrix U such that $B = U^{-1} A\, U$.

$$\text{(a) } A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix} \qquad \text{(b) } A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}.$$

5. Give examples (*not* from the text or lecture notes) of 2×2 matrices A in Jordan normal form such that the origin is a (a) sink; (b) source; (c) saddle; (d) center; (e) none of the above (i.e. not (a)–(d)) for the ODE $\dot{x} = A\,x$. Sketch the phase portrait, compute $\exp(t\,A)$, and sketch the image of the unit square under $\exp(t_*\,A)$ at some time $t_* > 0$ for each of your examples.

6. Consider the system

$$\dot{x} = x(\epsilon_1 - \sigma_1 x - \alpha_1 y) \qquad \dot{y} = y(\epsilon_2 - \sigma_2 y - \alpha_2 x),$$

   all constants positive, which describes two competing populations of fish. Identify and analyse the equilibria, and describe the long term behavior of the system, in each of the following three cases:

   (a) $\dfrac{\epsilon_2}{\alpha_2} > \dfrac{\epsilon_1}{\sigma_1}$ and $\dfrac{\epsilon_2}{\sigma_2} > \dfrac{\epsilon_1}{\alpha_1}$
   (b) $\dfrac{\epsilon_1}{\alpha_1} > \dfrac{\epsilon_2}{\sigma_2}$ and $\dfrac{\epsilon_1}{\sigma_1} > \dfrac{\epsilon_2}{\alpha_2}$
   (c) $\dfrac{\epsilon_2}{\alpha_2} > \dfrac{\epsilon_1}{\sigma_1}$ and $\dfrac{\epsilon_1}{\alpha_1} > \dfrac{\epsilon_2}{\sigma_2}$.

7. Give an example of a nontrivial ODE such that if $x(t)$ is a solution satisfying $x(0) = x_0$ and $y(t)$ is a solution satisfying $y(0) = y_0$, then $z(t) = x(t) + y(t)$ is a solution satisfying $z(0) = x_0 + y_0$. Give an example of an ODE for which this is *not* true.

8. Find the equilibria of the following systems and determine their stability type (sink, source, etc.); using your results, match the systems to the (incomplete) phase portraits shown below. Add the equilibria to the portraits.

$$\text{(a)}\quad \begin{aligned}\dot{x} &= 1 - y^2\\ \dot{y} &= x + 2\,y\end{aligned} \qquad \text{(b)}\quad \begin{aligned}\dot{x} &= x - y - x^2 + x\,y\\ \dot{y} &= -y - x^2\end{aligned} \qquad \text{(c)}\quad \begin{aligned}\dot{x} &= x - 2\,y\\ \dot{y} &= 4\,x - x^3\end{aligned}\ .$$

If you want more matrix exponentiation practice, compute $\exp(t\,A)$ for the linearizations of each of these systems at the equilibria.