Beating Plots

■ Definitions

In[124]:=
```
showbeat[q_, ω_, v_, tf_] := Plot[(Cos[ω t] - Cos[Sqrt[q] t])/(q - ω^2) + v/Sqrt[q] Sin[Sqrt[q] t], {t, 0, tf}]
```

$$\text{showbeat}[q\_, \omega\_, v\_, tf\_] := \text{Plot}\left[\frac{\text{Cos}[\omega\, t] - \text{Cos}\left[\sqrt{q}\; t\right]}{q - \omega^2} + \frac{v}{\sqrt{q}} \text{Sin}\left[\sqrt{q}\; t\right], \{t, 0, tf\}\right]$$

In[112]:=

$$\text{showpair}[q\_, \omega\_, tf\_] := \text{Plot}\left[\left\{\frac{\text{Cos}[\omega\, t] - \text{Cos}\left[\sqrt{q}\; t\right]}{q - \omega^2}, \text{Cos}[\omega\, t], \text{Cos}\left[\sqrt{q}\; t\right]\right\}, \{t, 0, tf\}\right]$$

In[125]:=

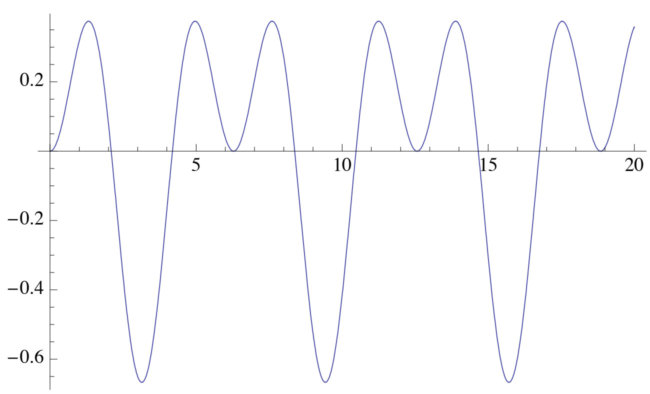
$$\text{envelope}[q\_, \omega\_, tf\_] := \text{Plot}\left[\{\#, -\#\} \,\&\left[\frac{2}{q - \omega^2} \text{Sin}\left[\frac{\sqrt{q} - \omega}{2} t\right]\right], \{t, 0, tf\}\right]$$

In[126]:=
```
wrap[q_, ω_, tf_] := Show[{envelope[q, ω, tf], showbeat[q, ω, 0, tf]}]
```

■ Sample plots

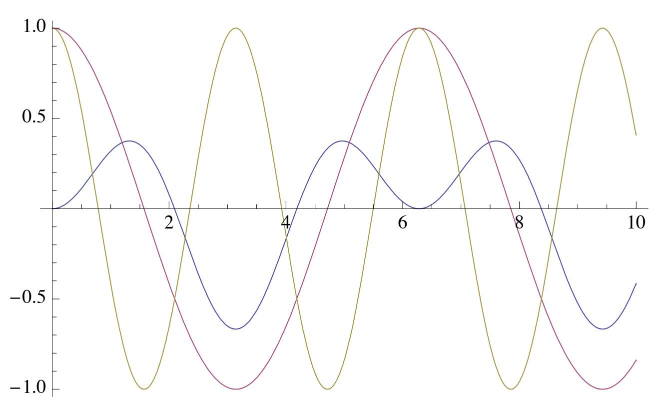
■ q = 4, $\omega = 1$, $\frac{\omega}{\sqrt{q}} = \frac{1}{2}$

```
showbeat[4, 1, 0, 20]
```

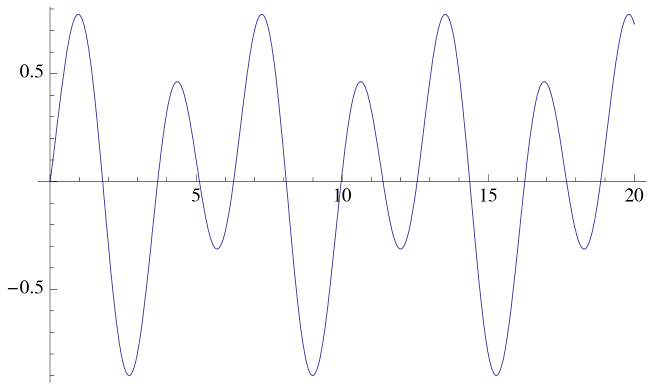


In[113]:=
```
showpair[4, 1, 10]
```

Out[113]=

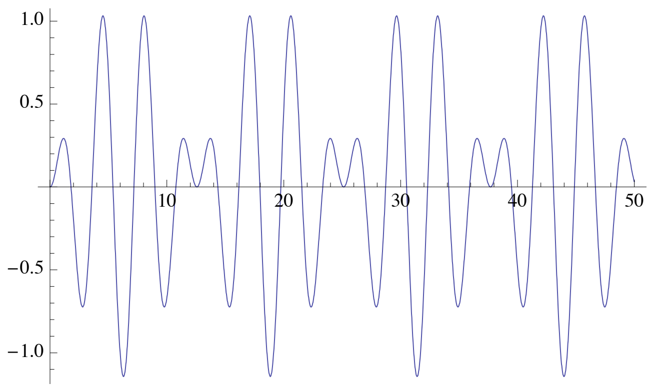


```
showbeat[4, 1, 1, 20]
```



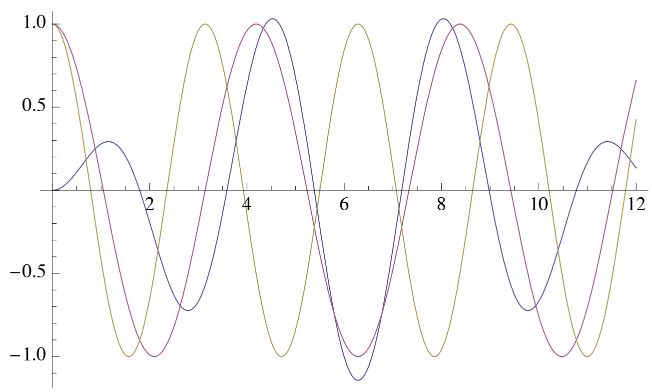
- **q = 4,** $\omega = \frac{3}{2}$, $\frac{\omega}{\sqrt{q}} = \frac{3}{4}$

```
showbeat[4, 1.5, 0, 50]
```

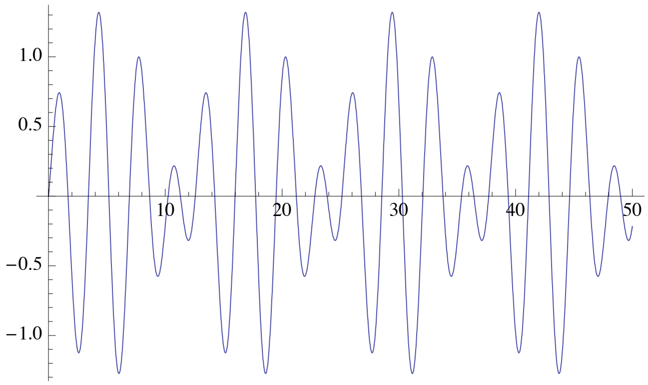


In[114]:=
```
showpair[4, 1.5, 12]
```

Out[114]=



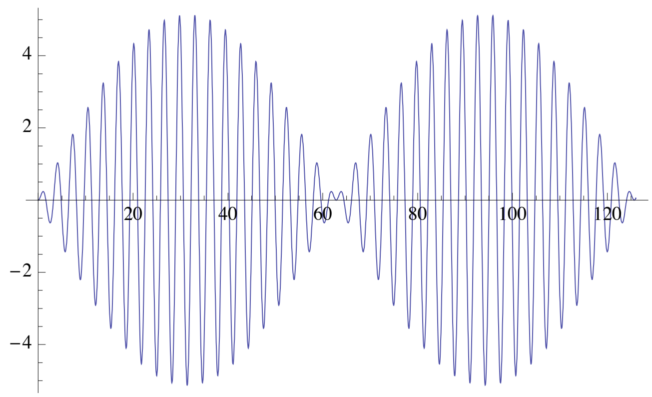
```
showbeat[4, 1.5, 1, 50]
```



- **$q = 4$, $\omega = 1.9$, $\frac{\omega}{\sqrt{q}} = .95$, $\frac{\sqrt{q} - \omega}{2} = .05$ (period of ' wrap ' is $\frac{2\pi}{.05}$, approx 126), $\frac{2}{q - \omega^2}$ is approx 5**

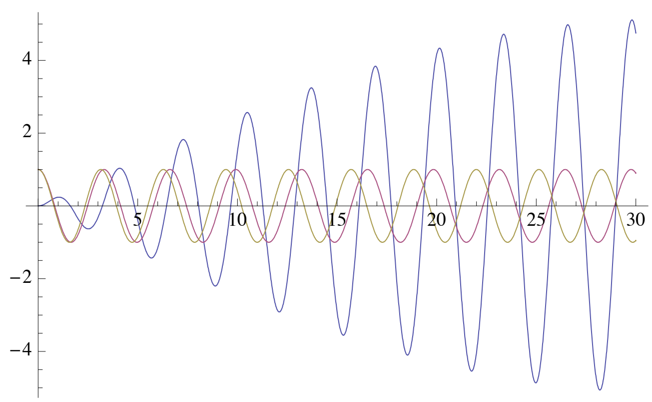
In[132]:= `showbeat[4, 1.9, 0, 126]`

Out[132]=



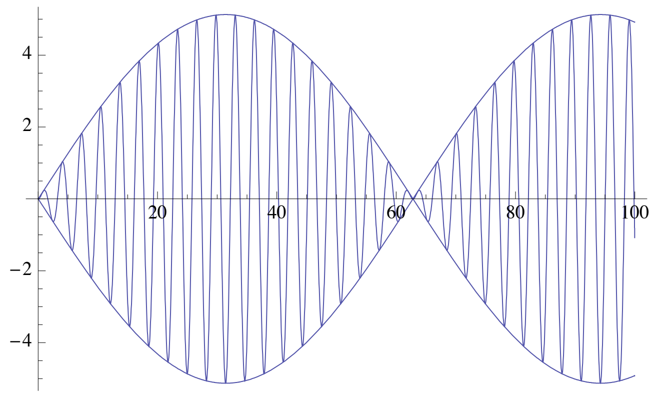
In[116]:= `showpair[4, 1.9, 30]`

Out[116]=



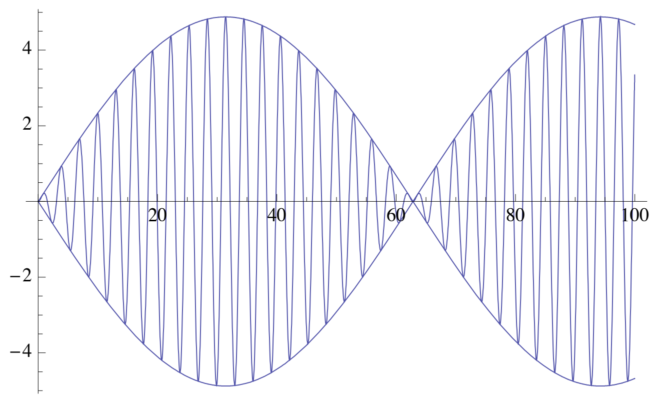
In[127]:= **wrap[4, 1.9, 100]**

Out[127]=

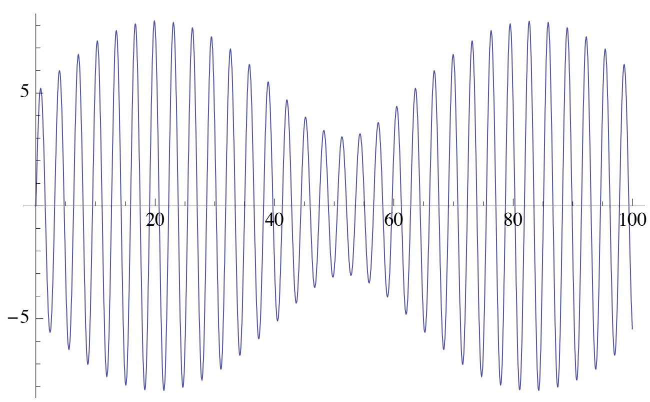


In[128]:= **wrap[4, 2.1, 100]**

Out[128]=

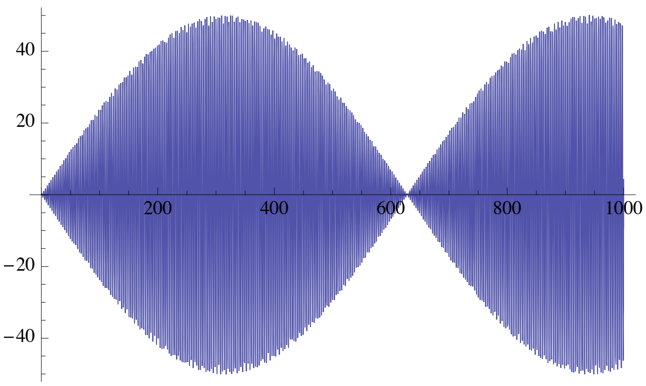


showbeat[4, 1.9, 10, 100]

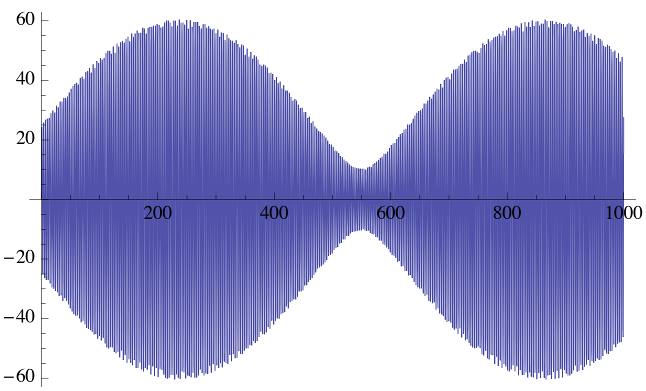


- q = 4, ω = 1.99, $\frac{\omega}{\sqrt{q}}$ = .995, $\frac{\sqrt{q}-\omega}{2}$=.005 (period of 'wrap' is $\frac{2\pi}{.005}$, approx 1257), $\frac{2}{q-\omega^2}$ is approx 50

```
showbeat[4, 1.99, 0, 1000]
```

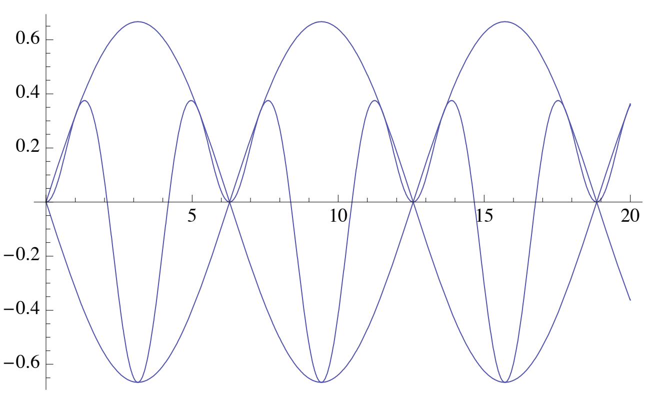



```
showbeat[4, 1.99, 50, 1000]
```



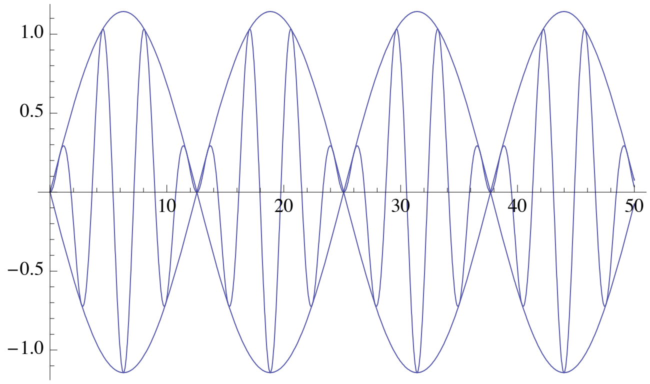
- q = 4, ω = 1, $\frac{\omega}{\sqrt{q}} = \frac{1}{2}$, $\frac{\sqrt{q}-\omega}{2} = \frac{1}{2}$(period of 'wrap' is hence 4π, approx. 12.5)

```
wrap[4, 1, 20]
```



- **q = 4, $\omega = \frac{3}{2}$, $\frac{\sqrt{q} - \omega}{2} = \frac{1}{4}$ (period of 'wrap' is hence 8π, approx. 25)**

```
wrap[4, 1.5, 50]
```



Partially coupled harmonic oscillators

In[143]:= `? JordanDecomposition`

JordanDecomposition[*m*] yields the Jordan decomposition of a square matrix *m*. The result is a list {*s*, *j*} where *s* is a similarity matrix and *j* is the Jordan canonical form of *m*. ≫

In[144]:= `JordanDecomposition[{{0, 1, 0, 0}, {-ω1^2, 0, 1, 0}, {0, 0, 0, 1}, {0, 0, -ω2^2, 0}}]`

Out[144]= $\left\{\left\{\left\{\frac{i}{\omega_1}, -\frac{i}{\omega_1}, -\frac{i}{\omega_2\left(-\omega_1^2+\omega_2^2\right)}, \frac{i}{\omega_2\left(-\omega_1^2+\omega_2^2\right)}\right\}, \left\{1, 1, \frac{1}{\omega_1^2-\omega_2^2}, \frac{1}{\omega_1^2-\omega_2^2}\right\}, \left\{0, 0, \frac{i}{\omega_2}, -\frac{i}{\omega_2}\right\}, \{0, 0, 1, 1\}\right\}, \{\{-i\,\omega_1, 0, 0, 0\}, \{0, i\,\omega_1, 0, 0\}, \{0, 0, -i\,\omega_2, 0\}, \{0, 0, 0, i\,\omega_2\}\}\right\}$

In[145]:= `MatrixForm[%[[2]]]`

Out[145]//MatrixForm=

$$\begin{pmatrix} -i\,\omega_1 & 0 & 0 & 0 \\ 0 & i\,\omega_1 & 0 & 0 \\ 0 & 0 & -i\,\omega_2 & 0 \\ 0 & 0 & 0 & i\,\omega_2 \end{pmatrix}$$

In[146]:= `MatrixForm[%%[[1]]]`

Out[146]//MatrixForm=

$$\begin{pmatrix} \frac{i}{\omega_1} & -\frac{i}{\omega_1} & -\frac{i}{\omega_2\left(-\omega_1^2+\omega_2^2\right)} & \frac{i}{\omega_2\left(-\omega_1^2+\omega_2^2\right)} \\ 1 & 1 & \frac{1}{\omega_1^2-\omega_2^2} & \frac{1}{\omega_1^2-\omega_2^2} \\ 0 & 0 & \frac{i}{\omega_2} & -\frac{i}{\omega_2} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

In[147]:= `MatrixExp[t {{0, 1, 0, 0}, {-ω1^2, 0, 1, 0}, {0, 0, 0, 1}, {0, 0, -ω2^2, 0}}] // MatrixForm`

Out[147]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[t\,\omega_1] & \frac{\text{Sin}[t\,\omega_1]}{\omega_1} & \frac{-\text{Cos}[t\,\omega_1]+\text{Cos}[t\,\omega_2]}{\omega_1^2-\omega_2^2} & \frac{\text{Sin}[t\,\omega_2]\,\omega_1-\text{Sin}[t\,\omega_1]\,\omega_2}{\omega_1\,\omega_2\,(\omega_1^2-\omega_2^2)} \\ -\text{Sin}[t\,\omega_1]\,\omega_1 & \text{Cos}[t\,\omega_1] & \frac{\text{Sin}[t\,\omega_1]\,\omega_1-\text{Sin}[t\,\omega_2]\,\omega_2}{\omega_1^2-\omega_2^2} & \frac{-\text{Cos}[t\,\omega_1]+\text{Cos}[t\,\omega_2]}{\omega_1^2-\omega_2^2} \\ 0 & 0 & \text{Cos}[t\,\omega_2] & \frac{\text{Sin}[t\,\omega_2]}{\omega_2} \\ 0 & 0 & -\text{Sin}[t\,\omega_2]\,\omega_2 & \text{Cos}[t\,\omega_2] \end{pmatrix}$$

In[148]:= `JordanDecomposition[{{0, 1, 0, 0}, {-ω^2, 0, 1, 0}, {0, 0, 0, 1}, {0, 0, -ω^2, 0}}]`

Out[148]= $\left\{\left\{\left\{\frac{i}{\omega}, \frac{1}{\omega^2}, -\frac{i}{\omega}, \frac{1}{\omega^2}\right\}, \{1, 0, 1, 0\}, \{0, 2, 0, 2\}, \{0, -2\,i\,\omega, 0, 2\,i\,\omega\}\right\},\right.$
$\left.\{\{-i\,\omega, 1, 0, 0\}, \{0, -i\,\omega, 0, 0\}, \{0, 0, i\,\omega, 1\}, \{0, 0, 0, i\,\omega\}\}\right\}$

In[151]:= `cfω = MatrixExp[t {{0, 1, 0, 0}, {-ω^2, 0, 1, 0}, {0, 0, 0, 1}, {0, 0, -ω^2, 0}}] // Expand`

Out[151]= $\left\{\left\{\frac{1}{2}e^{-i t \omega}+\frac{1}{2}e^{i t \omega}, \frac{i e^{-i t \omega}}{2\omega}-\frac{i e^{i t \omega}}{2\omega}, \frac{i e^{-i t \omega} t}{4\omega}-\frac{i e^{i t \omega} t}{4\omega}, \frac{i e^{-i t \omega}}{4\omega^3}-\frac{i e^{i t \omega}}{4\omega^3}-\frac{e^{-i t \omega} t}{4\omega^2}-\frac{e^{i t \omega} t}{4\omega^2}\right\},\right.$
$\left\{-\frac{1}{2} i e^{-i t \omega}\omega+\frac{1}{2} i e^{i t \omega}\omega, \frac{1}{2}e^{-i t \omega}+\frac{1}{2}e^{i t \omega},\right.$
$\left.\frac{1}{4}e^{-i t \omega} t+\frac{1}{4}e^{i t \omega} t+\frac{i e^{-i t \omega}}{4\omega}-\frac{i e^{i t \omega}}{4\omega}, \frac{i e^{-i t \omega} t}{4\omega}-\frac{i e^{i t \omega} t}{4\omega}\right\},$
$\left.\left\{0, 0, \frac{1}{2}e^{-i t \omega}+\frac{1}{2}e^{i t \omega}, \frac{i e^{-i t \omega}}{2\omega}-\frac{i e^{i t \omega}}{2\omega}\right\}, \left\{0, 0, -\frac{1}{2} i e^{-i t \omega}\omega+\frac{1}{2} i e^{i t \omega}\omega, \frac{1}{2}e^{-i t \omega}+\frac{1}{2}e^{i t \omega}\right\}\right\}$

In[149]:= **rfω =** $\left\{\left\{\text{Cos}[\omega t], \frac{\text{Sin}[\omega t]}{\omega}, \frac{t\,\text{Sin}[\omega t]}{2\omega}, \frac{\text{Sin}[\omega t]}{2\omega^3}-\frac{t\,\text{Cos}[\omega t]}{2\omega^2}\right\},\right.$
$\left\{-\omega\,\text{Sin}[\omega t], \text{Cos}[\omega t], \frac{t}{2}\text{Cos}[\omega t]+\frac{\text{Sin}[\omega t]}{2\omega}, \frac{t\,\text{Sin}[\omega t]}{2\omega}\right\},$
$\left.\left\{0, 0, \text{Cos}[\omega t], \frac{\text{Sin}[\omega t]}{\omega}\right\}, \{0, 0, -\omega\,\text{Sin}[\omega t], \text{Cos}[\omega t]\}\right\};$

In[152]:= `Factor[cfω - rfω, Trig → True]`

Out[152]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

In[153]:= `MatrixForm[rfω]`

Out[153]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[t\,\omega] & \frac{\text{Sin}[t\,\omega]}{\omega} & \frac{t\,\text{Sin}[t\,\omega]}{2\,\omega} & -\frac{t\,\text{Cos}[t\,\omega]}{2\,\omega^2}+\frac{\text{Sin}[t\,\omega]}{2\,\omega^3} \\ -\omega\,\text{Sin}[t\,\omega] & \text{Cos}[t\,\omega] & \frac{1}{2}t\,\text{Cos}[t\,\omega]+\frac{\text{Sin}[t\,\omega]}{2\,\omega} & \frac{t\,\text{Sin}[t\,\omega]}{2\,\omega} \\ 0 & 0 & \text{Cos}[t\,\omega] & \frac{\text{Sin}[t\,\omega]}{\omega} \\ 0 & 0 & -\omega\,\text{Sin}[t\,\omega] & \text{Cos}[t\,\omega] \end{pmatrix}$$

In[157]:= `Collect[Expand[rfω.{x0, v0, y0, u0}], {Cos[ω t], Sin[ω t]}] // MatrixForm`

Out[157]//MatrixForm=

$$\begin{pmatrix} \left(x0-\frac{t\,u0}{2\,\omega^2}\right)\text{Cos}[t\,\omega]+\left(\frac{u0}{2\,\omega^3}+\frac{v0}{\omega}+\frac{t\,y0}{2\,\omega}\right)\text{Sin}[t\,\omega] \\ \left(v0+\frac{t\,y0}{2}\right)\text{Cos}[t\,\omega]+\left(\frac{t\,u0}{2\,\omega}+\frac{y0}{2\,\omega}-x0\,\omega\right)\text{Sin}[t\,\omega] \\ y0\,\text{Cos}[t\,\omega]+\frac{u0\,\text{Sin}[t\,\omega]}{\omega} \\ u0\,\text{Cos}[t\,\omega]-y0\,\omega\,\text{Sin}[t\,\omega] \end{pmatrix}$$