Practice Problems for the Final Exam, Part 1

Math 105A Spring 2009

1. True/false. If true, prove it; if false, provide a counter-example.
   If A and B are disjoint closed subsets of a metric space X, then the set

   $$\{d(x,y)\,;\, x \in A \text{ and } y \in B\}$$

   must have a positive lower bound.

2. Does the sequence $\{\sqrt{n}\,(\sqrt{n+1} - \sqrt{n})\}$ converge? Justify your answer.

3. (a) Given a sequence $\{x_n\}$ of positive real numbers with $L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$, show that if $L > 1$, then the sequence diverges, and if $1 > L$, the sequence converges to 0.
   (b) Give an example of a convergent sequence of positive reals such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$.
   (c) Give an example of a divergent sequence of positive reals such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$.

4. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies $\lim_{x\to 0} f(x) = c$. Show that for any $s \in \mathbb{R}$, $s \neq 0$, $\lim_{x\to 0} f(s\,x) = c$. Is this still true if $s = 0$?

5. *Recall:* A choice of a subset $\mathcal{P}$ of a field $\mathcal{F}$ satisfying

   (a) for each $x \in \mathcal{F}$ exactly one of the following is true: $x \in \mathcal{P}$; $x = 0$; $-x \in \mathcal{P}$
   (b) $x \in \mathcal{P}$ and $y \in \mathcal{P}$ implies $x + y \in \mathcal{P}$
   (c) $x \in \mathcal{P}$ and $y \in \mathcal{P}$ implies $x\,y \in \mathcal{P}$

   makes $\mathcal{F}$ an ordered field, with $x < y \iff y - x \in \mathcal{P}$.

   Show that the set $\mathcal{P} := \mathcal{F} \cap \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers, determines an ordering on the field $\mathcal{F} := \{r + s\sqrt{2} : r, s \in \mathbb{Q}\}$, with addition and multiplication operations inherited from the reals.

6. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.

7. Let $I := [a,b]$ and let $f: I \to \mathbf{R}$ be bounded and continuous on I. Define $g: I \to \mathbf{R}$ by $g(x) := \sup\{f(t): a \leqslant t \leqslant x\}$ for $x \in I$. Prove that g is continuous continuous on I.

8. Let $g: \mathbf{R} \to \mathbf{R}$ satisfy the relation $g(x+y) = g(x)g(y)$ for all x, y in $\mathbf{R}$. Show that if g is continuous at $x = 0$, then g is continuous at every point of $\mathbf{R}$. Also if we have $g(a) = 0$ for some $a \in \mathbf{R}$, then $g(x) = 0$ for all $x \in \mathbf{R}$.

Difficulty: 1) Right answer should be clear; justification example may not be obvious
2) A little more subtle than likely exam question.
3) Exam-like
4) "
5) Be able to do this!!
6) Exam-level.
7, 8) A little harder/more subtle than exam questions.