Practice Problems, part 2, v. 2 Math 105A Spring 2009

1. Prove that the intersection of a finite number of open sets is open. (Don't just quote Rudin!)

2. Define the sequence $\{s_n\}$ by $s_1 := \sqrt{2}$ and $s_{n+1} := \sqrt{2 + s_n}$ for $n \in \mathbb{N}$. Show that s_n is irrational for every $n \in \mathbb{N}$.
   Hint: If $s_k \in \mathbb{Q}$, 'solve' $s_k = \sqrt{2 + s_{k-1}}$ for s_{k-1}.

3. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Show that if there exists $r < 1$ such that $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

4. Given $x \in [0, 1]$, set $a_n := \frac{(-1)^{3\,n}}{3^n} \sin(2\,\pi\,n\,x)$, $n \in \mathbb{N}$. Does the series $\Sigma\, a_n$ converge?

Remark on Problem 1 from Part 1: The 'trick' is that you need a moving target—you need pairs of points that get arbitrarily close to each other, but you can't let any point in A be a limit point of B or vice versa. In the example I gave at the review, the pairs $(0, 1/x) \in A =$ vertical axis and $(x, 1/x) \in B = \{(x, y) : x\,y = 1\}$ satisfy $d((0, 1/x), (x, 1/x)) = ||(x, 1/x) - (0, 1/x)|| = |x|$; as $|x| \to 0$, the points get closer horizontally, but move together vertically. I think the following is a 1-D example: let $A = \mathbb{N}$ and $B = f(\mathbb{N})$, where $f(n) := n + \frac{1}{n+1}$; $\lim_{n\to\infty} |f(n) - n| = \lim_{n\to\infty} \frac{1}{n+1} = 0$, so the infimum of the distances between points in A and B is 0.

Solution of Problem 7 from Part 1: Since f is continuous and $[a, b]$ is compact, f is uniformly continuous. Hence, given $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, i.e. $f(N_\delta(x)) \subseteq N_\epsilon(f(x))$.

Given $a \le x < y \le b$ satisfying $y - x < \delta$, $f(N_\delta(x)) \subseteq N_\epsilon(f(x))$ implies that $f([a, y]) \subseteq f([a, x]) \cup N_\epsilon(f(x))$. Hence

$$g(y) = \sup f([a, y]) \le \sup(f([a, x]) \cup N_\epsilon(f(x))) < \sup f([a, x]) + \epsilon = g(x) + \epsilon.$$

Since $x < y$ implies $g(x) \le g(y)$, we have $g(x) - \epsilon < g(y) < g(x) + \epsilon$, i.e. $|g(y) - g(x)| < \epsilon$, so g is uniformly continuous.