

Midterm (homework version, with commentary on Problems 2 and 5)

Math 105A Spring 2009

1. True/false. If true, prove it; if false, provide a counter-example.
 - (a) If one point is deleted from a closed subset of a metric space, the resulting set is still closed.
 - (b) If one point is deleted from an open subset of a metric space, the resulting set is still open.
 - (c) $\overline{\bigcap_{n=1}^{\infty} A_n} = \bigcap_{n=1}^{\infty} \overline{A_n}$ for any countable collection of subsets A_n of a metric space X .

2. Show that a choice of a subset \mathcal{P} of a field \mathcal{F} satisfying

- (a) for each $x \in \mathcal{F}$ exactly one of the following is true: $x \in \mathcal{P}$; $x = 0$; $-x \in \mathcal{P}$
- (b) $x \in \mathcal{P}$ and $y \in \mathcal{P}$ implies $x + y \in \mathcal{P}$
- (c) $x \in \mathcal{P}$ and $y \in \mathcal{P}$ implies $xy \in \mathcal{P}$

makes \mathcal{F} an ordered field, with $x < y \iff y - x \in \mathcal{P}$.

In other words, show that specifying the set of positive numbers in a field determines an ordering on that field that makes it an ordered field, provided that the set \mathcal{P} of positive numbers satisfies the three properties specified above. A choice of \mathcal{P} determines an ordering as follows: $x < y := (y - x \in \mathcal{P})$; i.e., the logical statement $x < y$ is assigned the value of the logical statement $(y - x \in \mathcal{P})$.

3.
 - (a) Show that if A and B are bounded subsets of \mathbb{R} , then $A \cup B$ is bounded.
 - (b) Show that if A and B are compact subsets of \mathbb{R} , then $A \cup B$ is compact.
4. The *distance* from a point p in a metric space X to a nonempty subset S of X is defined as

$$\text{dist}(p, S) := \inf\{d(p, s) : s \in S\}.$$

Show that $p \in \overline{S} \iff \text{dist}(p, S) = 0$.

5. Fill in the following outline of a proof that for any compact subset K of \mathbb{R} and any point $c \in \mathbb{R}$, there exists a point $a \in K$ such that $|c - a| = \text{dist}(c, K) = \inf\{|c - x| : x \in K\}$.
 - (a) For any $n \in \mathbb{N}$, there exists $x_n \in K$ satisfying $|c - x_n| < \text{dist}(c, K) + \frac{1}{n}$.
 - (b) If the set $\{x_n\}_{n \in \mathbb{N}}$ is finite, then $|c - x_j| = \text{dist}(c, K)$ for some $j \in \mathbb{N}$.
 - (c) If the set $\{x_n\}_{n \in \mathbb{N}}$ is infinite, it has a limit point $a \in K$ satisfying $|c - a| = \text{dist}(c, K)$.

Here you should prove (a), (b), and (c), and then use these pieces of information to prove the main result.