Midterm Solution Suggestions Math 105A Spring 2009

There are other good approaches to many of the problems

1. True/false. If true, prove it; if false, provide a counter-example.

   (a) If one point is deleted from a closed subset of a metric space, the resulting set is still closed.
   False. For example, $[0,1]$ *is closed, but* $[0,1]\backslash\{\frac{1}{2}\} = [0,\frac{1}{2}) \cup (\frac{1}{2},1]$ *is not.*

   (b) If one point is deleted from an open subset of a metric space, the resulting set is still open.
   True. We need to show that if A *is open and* $p \in A$, $B := A\backslash\{p\}$ *is open, i.e. that if* $q \in B$, *then* q *is an interior point of* B. *Since* A *is open,* q *is an interior point of* A, *so there exists* $r > 0$ *such that* $N_r(q) \subseteq A$. *Let* $s := \min\{r, d(p,q)\}$; *then* $N_s(q) \subseteq B$, *so* q *is an interior point of* B.

   (c) $\overline{\cap_{n=1}^{\infty} A_n} = \cap_{n=1}^{\infty} \overline{A_n}$ for any countable collection of subsets A_n of a metric space X.
   False. For example, let $A_n := (0,\frac{1}{n})$. $\cap_{n=1}^{\infty} A_n = \emptyset$, *and* $\overline{\emptyset} = \emptyset$. *On the other hand,* $\overline{A_n} = [0,\frac{1}{n}]$, *so* $\cap_{n=1}^{\infty}\overline{A_n} = \cap_{n=1}^{\infty}[0,\frac{1}{n}] = \{0\} \neq \emptyset$.

2. Show that a choice of a subset $\mathcal{P}$ of a field $\mathcal{F}$ satisfying

   (a) for each $x \in \mathcal{P}$ exactly one of the following is true: $x \in \mathcal{P}$; $x = 0$; $-x \in \mathcal{P}$

   (b) $x \in \mathcal{P}$ and $y \in \mathcal{P}$ implies $x + y \in \mathcal{P}$

   (c) $x \in \mathcal{P}$ and $y \in \mathcal{P}$ implies $x\,y \in \mathcal{P}$

   makes $\mathcal{F}$ an ordered field, with $x < y \iff y - x \in \mathcal{P}$.

   Given $x, y \in \mathcal{P}$, *exactly one of the following is true:*

   - $y - x \in \mathcal{P}$, *and hence* $y > x$
   - $y - x = 0$, *and hence* $y = x$
   - $x - y = -(y - x) \in \mathcal{P}$, *and hence* $x > y$.

   If $x, y, z \in \mathcal{F}$, $x < y$, *i.e.* $y - x \in \mathcal{P}$, *and* $y < z$, *i.e.* $z - y \in \mathcal{P}$, *then (b) implies that* $z - x = (z - y) + (y - x) \in \mathcal{P}$, *and hence* $z > x$. *Hence* $\mathcal{P}$ *determines an ordering on* $\mathcal{F}$.

   If $x, y, z \in \mathcal{F}$ *and* $y < z$, *i.e.* $z - y \in \mathcal{P}$, *then* $(x + z) - (x + y) = z - y \in \mathcal{P}$ *implies that* $x + z > x + y$. $x > 0 \iff x = x - 0 \in \mathcal{P}$ *(N.B. "*$\mathcal{P}$*" stands for positive); hence (c) implies that if* $x > 0$ *and* $y > 0$, *then* $x\,y > 0$. *Thus the ordering determined by* $\mathcal{P}$ *makes* $\mathcal{F}$ *an ordered field.*

3. (a) Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is bounded.
   A *bounded implies there exists* $x \in \mathbb{R}$ *and* $r > 0$ *such that* $A \subseteq N_r(x)$; *analogously,* $B \subseteq N_s(y)$ *for some* $y \in \mathbb{R}$ *and* $s > 0$. *Let* $M := \max\{r + |x|, s + |y|\}$. *Then* $z \in A \cup B$ *implies that either* $z \in A$, *and hence* $|z| \leq |z - x| + |x| < r + |x| \leq M$, *or* $z \in B$, *and hence* $|z| \leq |z - y| + |y| < s + |y| \leq M$. *Thus* $A \cup B \subseteq N_M(0)$.

   (b) Show that if A and B are compact subsets of $\mathbb{R}$, then $A \cup B$ is compact.
   The Heine-Borel Theorem implies that A *and* B *are both closed and bounded. Finite unions of closed sets are closed, so* $A \cup B$ *is closed, and part (a) implies that* $A \cup B$ *is bounded. Hence the Heine-Borel Theorem implies that* $A \cup B$ *is compact.*

4. The *distance* from a point p in a metric space X to a nonempty subset S of X is defined as

$$\text{dist}(p, S) := \inf\{d(p, s) : s \in S\}.$$

Show that $p \in \overline{S} \iff \text{dist}(p, S) = 0$.

If dist$(p, S) = 0$, that implies that for every $\epsilon > 0$, there exists $s \in S$ such that $d(p, s) < \epsilon$. (If not, ϵ would be a lower bound of the set $D := \{d(p, s) : s \in S\}$ and hence dist(p, S) would be at least ϵ.) Hence either $p \in S$ or p is a limit point of S (possibly both), so $p \in \overline{S}$.

If $p \in \overline{S}$, then either $p \in S$ or p is a limit point of S (possibly both). If $p \in S$, then $0 \in D\}$ and hence, since 0 is a lower bound of D, dist$(p, S) = 0$. If p is a limit point of S, for every $\epsilon > 0$, there exists $s \in S$ such that dist$(p, S) \leq d(p, s) < \epsilon$; this implies dist$(p, S) = 0$.

5. Fill in the following outline of a proof that for any compact subset K of $\mathbb{R}$ and any point $c \in \mathbb{R}$, there exists a point $a \in K$ such that $|c - a| = \text{dist}(c, K) = \inf\{|c - x| : x \in K\}$.

   (a) For any $n \in \mathbb{N}$, there exists $x_n \in K$ satisfying $|c - x_n| < \text{dist}(c, K) + \frac{1}{n}$.
   If not, dist$(c, K) + \frac{1}{n}$ would be a lower bound of $\{|c - x| : x \in K\}$, but dist(c, K) is the greatest lower bound of that set.

   (b) If the set $\{x_n\}_{n \in \mathbb{N}}$ is finite, then $|c - x_j| = \text{dist}(c, K)$ for some $j \in \mathbb{N}$.
   Assume $\{x_n\}_{n \in \mathbb{N}}$ contains a finite number of distinct elements $\{x_{n_1}, \ldots x_{n_k}\}$. dist$(c, K) \leq \min\{|c - x_{n_1}|, \ldots, |c - x_{n_k}|\} \leq$ dist$(c, K) + \frac{1}{n}$ for all $n \in \mathbb{N}$ implies $\min\{|c - x_{n_1}|, \ldots, |c - x_{n_k}|\} =$ dist(c, K). If the minimum is achieved at n_ℓ (not necessarily unique), let $j = n_\ell$.

   (c) If the set $\{x_n\}_{n \in \mathbb{N}}$ is infinite, it has a limit point $a \in K$ satisfying $|c - a| = \text{dist}(c, K)$.
   Any infinite subset of a compact set has a limit point; let a be a limit point of $\{x_n\}_{n \in \mathbb{N}}$. For any $\epsilon > 0$, the neighborhood $N_{\epsilon/2}(a)$ must contain infinitely many elements of $\{x_n\}_{n \in \mathbb{N}}$. Let $m \in \mathbb{N}$ be such that $\frac{1}{m} < \frac{\epsilon}{2}$, so that $n \geq m$ implies that $|c - x_n| < \frac{\epsilon}{2}$; since there are only $m - 1$ points in $\{x_n\}_{n \in \mathbb{N}}$ with index less than m, there are infinitely many x_n with index greater than or equal to m in $N_{\epsilon/2}(a)$.