Name (please print): **Total:**

1. **2.** **3.** **4.** **5.** **6.** **7.**

Final Math 105A Spring 2009

Instructions: Solve five of the seven problems; in the list at the top of this page, circle the numbers of the problems you want graded. You may do the remaining problems for extra credit; please indicate clearly which are your extra credit problems, if any.

You can use a two page 'crib sheet', i.e. a set of notes on two 8.5 × 11 sheets of paper. No other notes, references, etc. are allowed.

You must justify your answers. If you don't have time to complete a problem, briefly outline the approach that you think might work. If you're unsure about something, it's best to say so; it's much better to indicate that you're uncertain, and would check something if you could, than to give the impression that you think everything's fine when it isn't.

1. True/false. If true, prove it; if false, provide a counter-example.
   (a) The union of a countable collection of compact sets is compact.
   (b) If $\{a_n\}$ and $\{b_n\}$ are convergent sequences in $\mathbb{R}$ satisfying $a_n < b_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n$.
   (c) Endow the integers $\mathbb{Z}$ with the distance function $d(j,k) := |j-k|$. If there is a continuous map from a metric space X onto $\mathbb{Z}$, then X is not connected.
   (d) Every Cauchy sequence in a metric space X converges.
   (e) If $\{x_n\}$ and $\{y_n\}$ are divergent sequences, then the sequence $\{x_n + y_n\}$ diverges.

2. Show that $\mathcal{F} := \{r + s\sqrt{2} : r, s \in \mathbb{Q}\}$, with addition and multiplication operations inherited from the reals, is a field, and that $x < y \iff y - x \in \mathcal{P} := \{r + s\sqrt{2} \in \mathcal{F} : r - s\sqrt{2} \in \mathbb{R}^+\}$ determines an ordering on $\mathcal{F}$. (Here $\mathbb{R}^+$ denotes the set of positive real numbers, and the minus sign in the definition of $\mathcal{P}$ is not a typo.)

3. (a) Does the series $\sum \left(\frac{1}{2^n} - \frac{1}{3^n}\right)$ converge?
   (b) Does the series $\sum \frac{n}{1.01^n}$ converge?

4. (a) Given a subset A of $\mathbb{R}$ and a function $f : A \to \mathbb{R}$, show that the following are equivalent:
      i. f is not uniformly continuous on A
      ii. There exists an $\epsilon > 0$ such that for every $\delta > 0$ there is a point $x_\delta \in A$ such that $f(N_\delta(x_\delta))$ is not a subset of $N_\epsilon(f(x_\delta))$.
   (b) Give an example of a function that is continuous everywhere on its domain, but is not uniformly continuous. Identify an $\epsilon > 0$ and points x_δ as in part a.ii. for your example.

5. Show that an infinite subset E of a compact set K must have a limit point in K. Work directly from the relevant definitions—don't quote the result from Rudin.

6. Using the following properties of cos and sin, show that cos and sin are continuous on $\mathbb{R}$:

$$|\sin x| \le |x|$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cos x - \cos y = 2 \sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y-x}{2}\right)$$

$$\sin x - \sin y = 2 \cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

   Are they uniformly continuous?

7. (a) True/false. Justify your answers!
   i. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f'(x)$ exists for all nonzero x, and $\lim_{x\to 0} f'(x)$ exists, then f is differentiable at 0.
   ii. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ if $x < 0$ and $f(x) = x + x^2$ if $x \ge 0$ has second derivative equal to 2 on all of $\mathbb{R}$.

   (b) Show that if $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(y) - f(x)| \le |y - x|^2$ for all $x, y \in \mathbb{R}$, then f is constant.

Survey questions:
(If you don't have time to answer these now, *please* take a few minutes to let me know by email.)

Which of the following did you find most useful: section, office hours, 'study hall', Learning Support Services sessions, Instant Messaging study group.

Would you prefer to do more independent reading in the text, with, e.g., one lecture slot per week replaced by an instructor-led study session, with students working in small groups?

What do you think is the most effective use of TAs in upper division courses (given the loss of graders): sections, office hours, grading homework, preparing homework solutions that will be posted online.

Other suggestions, comments regarding course resources?

Have a great summer!