Sherman, M. A. (1976). Adjectival negation and the comprehension of multiply-negated sentences. *Journal of Verbal Learning & Verbal Behavior, 15,* 143–157.
Slobin, D. I, (1965). Grammatical transformations and speed of understanding. *Journal of Verbal Learning & Verbal Behavior, 5,* 219–227.
Trabasso, T., Rollin, H., & Shaughnessey, E. (1971). Storage and verification shapes in processing concepts. *Cognitive Psychology, 2,* 231–289.
Tversky, A., & Kahneman, D. (1977). Judgment under uncertainty: Heuristics and biases. In P. N. Johnson-Laird & P. C. Wason (Eds.), *Thinking: Readings in cognitive science* (pp. 326–337). Cambridge: Cambridge University Press.
Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter, J. M. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 39–59). Hillsdale, NJ: Lawrence Erlbaum Associates.
Wason, P. C. (1959). The processing of positive and negative information. *Quantitative Journal of Experimental Psychology, 11,* 92–107.
Wason, P. C. (1961). Response to affirmative and negative binary statements. *Journal of Psychology, 52,* 133–142.
Winograd, T. (1972). *Understanding natural language.* New York: Academic Press.

Chapter 13

Linguistic Features of Mathematical Problem Solving: *Insights and Applications*

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INTRODUCTION

Language Proficiency and Mathematics Achievement

Students whose home language is not English must acquire a substantial level of English proficiency to be able to participate effectively in U.S. schools. Cummins (1981) postulates that there exists a minimal level of linguistic competence—a threshold—that a student must attain to function effectively in cognitively demanding, academic tasks. This threshold of cognitive academic language proficiency (CALP) can take between 5 and 7 years to develop in a student's second language.

Burns, Gerace, Mestre, and Robinson (1983), expanding this notion, have proposed a technical threshold to explain the strong positive correlation they found between the English language proficiency of Hispanic college students enrolled in technical classes and their performance on mathematics tests. They argue that students who lack language skills or the cognitive skills to solve technical problems, or both, have not yet reached the level required to comfortably read and solve technical problems. They believe that materials and teaching methods need to be developed that will assist students in attaining this threshold so that they can participate effectively in the cognitively demanding tasks required in mathematics and science classes.

Dawe (1984) suggests an interesting perspective on the role that language plays in the performance of cognitively demanding tasks in mathematics. He entertains the hypothesis that students must reach a threshold level of proficiency in cognitive academic mathematics proficiency (CAMP). CAMP consists of cognitive knowledge (mathematical concepts and how they are applied) embedded in a language specifically structured to express that knowledge. The threshold level for CAMP, which students must achieve to perform math tasks, consists of proficiency both in mathematics and in math language. This tentative proposal lends support to Mestre's (1986) suggestion that math instruction for limited-English-proficient students (if not for all students) should follow an approach that integrates rather than separates math skills and language skills.

A growing body of research suggests a close relationship between language proficiency and mathematics achievement. Studies with monolingual English speakers have revealed a high positive correlation between mathematics achievement and English reading ability (Aiken, 1971). Duran (1979) found a similar positive correlation between the reading comprehension skills of Puerto Rican college students and their performance on deductive reasoning problems in English and Spanish, with a similar pattern across both languages. Mestre (1981) found a strong correlation between language proficiency and math performance among college Hispanic engineering students as did Cossio (1978). Cuevas (1984) has shown that language is a factor both in the learning of mathematics and in the assessment of mathematics achievement. Further, several researchers have found that language minority students frequently do not understand the language used to present math test problems (Crandall, Dale, Rhodes, & Spanos, in press; DeAvila & Havassy 1974; Moreno 1970; Ramirez & Gonzalez, 1972).

Thorndike (1912) said, "Our measurement of arithmetic is a measure of two things: sheer mathematical knowledge on the one hand, and acquaintance with language on the other." Language skills are the vehicles through which students learn, apply, and are tested on math concepts and skills. Unfortunately, the language of mathematics is often too difficult for many students. Consider the following elementary algebra problem:

> Find a number such that seven less than the number is equal to twice the number minus 23.

Although the solution to the problem is straightforward once the proper equation for the solution has been derived, the linguistic skills required to reach that point are rather sophisticated. First, one must understand that *such that* relates *seven less than the number* to *twice the number minus 23*. It is also necessary to understand that *a number* and the subsequent *the number* (which is repeated twice) refer to the same number. Finally, the phrase *seven less than the number* is syntactically confusing. It often leads the student to write $7 - n$, when the reverse, $n - 7$, is required. The problem also assumes that the student understands that the phrase *is equal to* signals an equation. Gerace and Mestre (1983) have demonstrated that these types of constructions cause difficulties for ninth-

grade algebra students with limited English proficiency. Crandall et al. (in press) have found similar problems among college developmental algebra students.

Halliday (1975) has suggested that mathematics comprises a unique linguistic register with special features that must be mastered by students of mathematics and mathematics-related disciplines. He defines a linguistic register as "a variety of language that is oriented to a particular context, to a certain type of activity, involving certain groups of people, with a certain type of activity, involving certain groups of people, with a certain rhetorical force" (p. 5). Thus, the mathematics register would be the variety of language oriented to mathematics activities and would comprise the various linguistic forms, their meanings and uses, that appear in the context of these activities.

Despite the evidence and conjectures cited above, limited-English-proficient students are traditionally "mainstreamed" into mathematics classes before they are placed into other academic content classes on the invalid assumption that math is "language independent." To the contrary, students would benefit from an approach that simultaneously taught language and mathematics content.

An Applied Research Project

This close and constant relation between language proficiency and mathematics achievement has motivated an applied research project to (a) investigate linguistic features that pose difficulties in understanding and solving algebra problems and (b) develop materials that would address these linguistic difficulties while increasing students' understanding of basic algebra concepts. Working collaboratively with mathematics departments at three colleges (Miami-Dade Community College, Northern Virginia Community College, and Metropolitan State College), we have developed an approach and created materials to assist students in becoming proficient in using the language of math (Crandall, Dale, Rhodes, & Spanos, 1987). In first year algebra, students review basic math and begin working with increasingly abstract and symbolic language. For that reason, and because algebra is a gatekeeping course for all scientific and technical disciplines, attention was focused on developmental algebra.

Data collection procedures at the three colleges included analyzing texts and tests, and most important, recording student discussions as they worked together to solve math problems. Forty-six developmental and college algebra students, some Hispanic, some limited-English-proficient, and some native English-speaking, were interviewed, usually in groups of two or three students. We asked them to complete general information questionnaires and then to talk together as they attempted to solve a series of algebra problems. These think-aloud sessions, which lasted about an hour each, were audio-taped and later transcribed. The data from these problem-solving sessions have been a particularly important source of information on the difficulties students face in working through arithmetic and algebra word problems and on the particular linguistic features that are most problematic.

When students were first asked, individually, to explain what they were thinking about and how they were approaching standard algebra problems, they were often unable to do so, since much of their difficulty with math resided in their inability to use math language or to verbalize their thinking and problem-solving strategies. However, when placed in groups of two or three students and asked to cooperatively solve a problem, such verbalization came much more naturally. Students suggested ways in which the problem might be understood or stated in mathematical terms for solution or they asked questions of other students.

A LINGUISTIC MODEL FOR CATEGORIZING FEATURES OF THE MATHEMATICS REGISTER

As a result of the interviews, text and test analyses, and problem-solving sessions, we were able to identify a number of types of linguistic difficulties students face in dealing with the mathematics register. In this section, we cite specific syntactic, semantic, and pragmatic features from our research that caused difficulties for beginning algebra students. In the next section, we discuss the broader problem-solving contexts in which such difficulties have been observed to arise.

It is useful in this regard to appeal to an important trichotomy proposed by Morris (1955) and adopted by Carnap (1955) to categorize the linguistic features that attend particular scientific domains. In his seminal work in semiotics (the systematic study of linguistic and nonlinguistic signs) Morris (1955) distinguished between the following three sub-studies:

syntactics: the study of how linguistic signs, or symbols, behave in relation to each other, e.g., the formal relationship that obtains between active and passive verb forms in English;

semantics: the study of how linguistic signs behave in relation to the objects or concepts they refer to (their denotations) or their senses (their connotations), e.g., the relationship between the English word *star* and the numerous heavenly and earthly objects to which it may refer and the several senses it may have;

pragmatics: the study of how linguistic signs are used and interpreted by speakers of natural languages in specific contexts of use, e.g., the relationship in English between the words "I promise" and a speaker's intentions to perform a future action or a hearer's expectations that the action will be performed.

In applying these definitions to the signs and symbols that characterize the mathematics register, we must come to grips with both the notational forms created for use by mathematicians and the English equivalents of these forms as they appear in mathematical discourse, including student problem sets. Thus, the source of many of the difficulties discussed in this section can be traced at the outset to the complex interplay between syntactic, semantic, and pragmatic features that occurs when students attempt to verbalize or interpret mathematical rules and concepts in English. Using this categorization of features, we can cite specific problematic features in the outline given in Table 13.1.

Discussion of Syntactic Features

Knight and Hargis (1977) have pointed out that comparative structures are especially useful and widespread in mathematical discourse. Unfortunately for many language minority students, the English comparative system can be rather complex. Structures such as *greater than, less than, n times as much as, as . . . as* are often confusing at the syntactic level because they require that the student master complex patterns that relate to specific meanings in a variety of ways.

Consider, for example, the fact that the following sentences are paraphrases of one another:

1. Triangle A is as large as Triangle B.
2. Triangles A and B are equal in size.
3. Triangle A and Triangle B are the same in size.

The same problem (cited by Munro, 1979) exists with the syntactic patterns required by prepositional phrases and the passive voice. Consider, for example, sentences 4–6:

4. Four divided into nine equals nine-fourths.
5. Nine divided by four equals nine-fourths.
6. If nine is divided by four, nine-fourths results.

As with the comparative structures in 1–3, sentences 4–6 are paraphrases that require a rather advanced facility with two-word verbs ending in prepositional particles and corresponding passives. The potential for confusion becomes evident when one considers that textbook writers and instructors are apt to employ a variety of patterns in their exercises and lectures that, although stylistically desirable, are beyond the competence of a sizable number of students, both native and foreign.

A further syntactic difficulty is related to the lack of a regular one-to-one

TABLE 13.1
Syntactic, Semantic and Pragmatic Features of the Mathematics Register

SYNTACTIC FEATURES

I. COMPARATIVES

greater than/less than	as in	all numbers greater than 4
n times as much as	as in	Hilda earns six times as much as I do. Hilda earns $40,000 a year. What do I earn?
as. . .as	as in	Wendy is as old as Jack. Jack is three years older than Frank. Frank is 25. How old is Wendy?

II. PREPOSITIONS

divided *into*	as in	four (divided) into nine (9/4 or 9 ÷ 4)
divided *by*	as in	four divided by nine (4/9 or 4 ÷ 9)
by	as in	two is multiplied by itself three times (multiplication) vs. x exceeds two by seven (addition)

III. PASSIVE VOICE
as in x is defined to be greater than or equal to zero.
When 15 *is added* to a number, the result is 21. What is the number?

IV. REVERSAL ERRORS
Examples: The number a is five less than the number b.
correct equation: $a = b - 5$
incorrect equation: $a = 5 - b$ or $a - 5 = b$

There are five times as many students as professors in the mathematics department.
correct equation: $5p = s$
incorrect equation: $5s = p$

V. LOGICAL CONNECTORS

if. . .then	as in	If a is positive then $-a$ is negative.
if and only if	as in	$a + b = c$ if and only if $b + a = c$.
given that	as in	Given that $a = 0$, $a \times b = 0$.

SEMANTIC FEATURES

I. LEXICAL
A. NEW TECHNICAL VOCABULARY

additive inverse	coefficient	denominator
binomial	monomial	polynomial

B. NATURAL LANGUAGE VOCABULARY WHICH HAS A DIFFERENT MEANING IN MATHEMATICS

square	rational	irrational
power	equality	inequality

C. COMPLEX STRINGS OF WORDS OR PHRASES
least common denominator
negative exponent
the quantity, y + 3, squared

(*Continued*)

D. SYNONYMOUS WORDS AND PHRASES
For addition: add, plus, combine, sum, more than, and increase by
For subtraction: subtract, minus, differ(ence), less than, and decreased by

E. SYMBOLS AND MATHEMATICAL NOTATION AS "VOCABULARY"

$=$	$>$	$\geqslant$	( )
$\sim$	$<$	$\leqslant$	[ ]

II. REFERENTIAL
A. ARTICLES/PRE-MODIFIERS

a number. . .the number	as in	Five times a number is two more than ten times the number.
one number. . .another number	as in	One number is ten times another number. If the first number is 7, find the second number.

B. VARIABLES
Example: There are five times as many apples as bananas in the fruit bowl.
correct equation: $5b = a$, where b refers to the *number* of bananas and a refers to the *number* of apples

III. VAGUENESS IN PROBLEMS AND DIRECTIONS
Example: Food expenses take 26% of the average family's income. A family makes $700 a month. How much is spent on food?
(That month? In a year? What is being asked for??)

IV. SIMILAR TERMS, DIFFERENT FUNCTIONS

less	vs.	less than
the square	vs.	the square root
divided by	vs.	divided into
multiply by	vs.	increased by

PRAGMATIC FEATURES

I. EPISTEMOLOGICAL
A. Lack of Experience or Knowledge
e.g., market-place concepts, e.g., discounts, cost, selling price, markup, wholesale, retail, sales tax rates
B. Restricted Experience or Knowledge
e.g., attempt to substitute known quantities like local tax rates for tax rates referred to in word problems
C. Conflicting Experience or Knowledge
e.g., inability to solve for tax rate because in practical experience this is a given fact
D. Contradictory Experience or Knowledge
e.g., discrepancy between the way sales tax is rounded off on actual sales tax charts and the way it is rounded off conventionally

II. TEXTUAL
A. Lack of Real Life Objects or Activities (Realia) in Math Curricula
B. Lack of Natural Interaction

correspondence between the strings of mathematical symbols in numerical expressions and equations and the strings of words used in their natural language counterparts. For example, although sentences 4–6 above are all symbolically equivalent to the equation $9/4 = 9/4$, students often attempt to duplicate the surface word order in rendering the equations into symbolic notation. Such an attempt would result in a reversal error for sentence 4, since the word order might lead one to write $4/9 = 9/4$. Similar errors noted by Crandall et al. (in press) are evident in sentence pairs 7a, 7b and 8a, 8b below:

7a. The number a is five less than the number b.
7b. $a = 5 - b$ (instead of $a = b - 5$)

8a. There are three times as many girls as boys at this university.
8b. $3g = b$ (instead of $3b = g$)

Logical connectors, which are "words or phrases which carry out the function of marking a logical relationship between two or more basic linguistic structures (and) serve a semantic, cohesive function indicating the nature of the relationship between parts of a text" (Kessler, Quinn, & Hayes, 1986, p. 14), are widely used in mathematics texts to develop and concatenate mathematical structures. Some of these connectors include *if . . . then, if and only if, because, that is, for example, such that, but, consequently, and either . . . or.*

When students read mathematics texts, they must be able to recognize logical connectors and the situations in which they appear. They must know which situation is signaled—similarity, contradiction, cause/effect or reason/result, chronological or logical sequence. On the level of syntax, they have to know where logical connectors appear in a sentence (clause initial, medial, or final) and they must be aware that some connectors can only appear in one position, but others can appear in two or all three positions. Moreover, a change in position can signal a change in meaning.

Examples abound in math texts where logical connectors are used to introduce definitions and properties—concepts that students must understand and apply to solve problems. These connectors most often appear in complex statements using both words and symbols, as in the following examples from Dolciani and Wooton (1970, p. 77):

1. If a is a positive number, then $-a$ is a negative number;
   if *a is a negative number, then* $-a$ is a positive number;
   if a is 0, then $-a$ is 0.
2. The opposite of $-a$ is a; that is, $-(-a) = a$.

It is not hard to imagine that native English-speaking students would have difficulty piecing together the logical statements in this section of text. And it is easy to see how non-native students might have even more difficulty when reading it in English.

Discussion of Semantic Features

Problems involving syntactic patterns are compounded by semantic phenomena ranging from the meanings of isolated vocabulary items (e.g., two-word verbs like *divided into*), to paraphrase relations between both natural language and formal language expressions (e.g., four divided into nine; nine divided by four; $9 \div 4$), to the referents of variables (e.g., $5g = b$ where g and b represent numbers of girls and boys), to inferences that may be drawn from linkages established by logical connectors (e.g., if . . . then sentences). Although students may have substantial practice with the syntactic patterns of English, there is no guarantee that they have acquired the denotative, connotative and conceptual network (the semantic patterns) that accompanies the individual words, phrases, and sentences common in mathematical discourse.

Lexical Items (Vocabulary). Mathematics vocabulary includes a set of words that are specific to mathematics, which must be learned new by most students. These words—*divisor, denominator, quotient,* and *coefficient*—are relatively easy to learn. However, the mathematics register also includes everyday vocabulary items that have a different meaning in mathematics. Words such as *equal, rational, irrational, column,* and *table* have to be learned again, this time in a math context, since they have a specialized meaning in mathematics.

In addition to isolated vocabulary items, complex strings of words or phrases are used. If is often the case that the combination of two or more mathematical concepts are put together to form a new concept, thereby compounding the task of comprehending the words. The phrases, *least common multiple, negative exponent,* and even something apparently simple like *a quarter of the apples* are good examples of the complexity of mathematical phrases.

A subtler and much more difficult aspect of math vocabulary involves the many ways in which the same math operation can be signaled. As students progress through the hierarchy of math skills, manipulation of this vocabulary becomes crucial for understanding teacher explanations in class and for solving word problems. Crandall et al. (in press) have identified groups of lexical items in beginning algebra that signal that certain operations should be undertaken. For example, addition can be signaled by any of these words:

add	and
plus	sum
combine	increased by

Similarly, subtraction can be signaled by these words:

subtract from	minus
decreased by	differ
less	less than

It is important to note that the meanings of such terms are tied to specific operations. For example, students learn early in their mathematics education that the word *by* signals multiplication as in the expression *three multiplied by 10.* However, when later faced with algebraic expressions such as *a number increased by 10,* where *by* is part of an expression that signals addition, they are confused. Prepositions, in general, and the relationships they indicate are critical lexical items in the math register, items that cause a great deal of confusion. Additional examples include *divided by* versus *divided into* or *toward* and *to* in word problems involving distance (Crandall et al., in press; p. 11).

In addition to words and phrases particular to the math register, students must also learn the set of notational symbols used in expressing mathematical concepts and processes. The more advanced the mathematics, the greater the number of symbols and the more conceptually dense their meanings. Hence students seeing symbols such as $<$ and $>$ and parentheses () and brackets [] must learn how to relate symbol to mathematical concept or process (morst likely couched in math language) and then translate these into everyday language in order to express the mathematical ideas embedded in the symbols.

It is also important to remember that there are some symbols that have different meanings depending upon the part of the world in which they are used. In a number of Latin American countries, for example, a comma is used to separate whole numbers from decimals—a function of the decimal point in the United States—and the decimal point is used with whole numbers to separate hundreds from thousands, hundred thousands from millions, and so forth (e.g., 4.500,36 instead of 4,500.36).

Inferential Meaning and Reference. Correctly manipulating the special vocabulary and phrases and the word order found in mathematics discourse is intricately tied to the ability to infer the correct mathematical meaning from the language. Making such inferences often depends on the language user's knowledge of how reference is indicated. In algebra, for example, the correct solution for word problems often hinges upon identifying key words and then knowing the other words in the problem to which those key words refer. For example, in a problem such as

> Five times a number is six more than two times the number. Find the number.

students must realize that *a number* and *the number* refer to the same quantity.

In the problem,

> The product of two numbers is 90. If the first number is 10 times the other, find the number.

students must know that they are dealing with two numbers. Furthermore, they must know that the wording of the problem links each number with unique information; for example, the referent of *the first number* and the information given about it (*10 times the other*). Moreover, students must know that the translation from the words of the problem to the symbolic representation of the solution equation will be based on only one variable, and that each of the two numbers will be expressed in terms of that one variable.

A common mistake for students who realize the problem is about two numbers is to write solution equations with two *different* variables. They see no relationship between the two numbers described in the problem and consequently cannot write an equation using only one variable to represent the two numbers.

The above examples point to another referential feature of the language of mathematics: the identification of variables. Identifying the referents of variables is essential to correctly translating the words of a problem into the symbols of its solution equation. Variables stand for the *number* of persons or things, not for the persons or things themselves. The classic "students and professors" type of word problem illustrates this point:

> There are five times as many students as professors in the mathematics department. Write an equation that represents this statement. Use s to represent the number of students and p to represent the number of professors.

Many students write the following incorrect equation, $5s = p$, which follows the literal word order of the natural language sentence and uses s to represent "students" and p to represent "professors." The correct equation is: $5p = s$, which can be determined only if students know that the variable s (or any other variable they choose to use) must represent the NUMBER of students and that the variable p must represent the NUMBER of professors. This "reversal error" has been the subject of investigation by a number of researchers interested in the language of mathematics, including Clement (1982) and Mestre, Gerace, and Lochhead (1982). Firsching (1985) points out that recurring reversal errors could result from students' previous, repeated exposure in "beginning" level mathematics to word problems whose solution equations consistently require a translation based on a one-to-one correspondence between words and symbols. Intensively trained to solve word problems this way, they incorrectly assume that all problems can be solved in the same manner.

Discussion of Pragmatic Features

Syntactic patterns and semantic relationships occur in the context of larger stretches of discourse, i.e., the expository prose and exercises found in textbooks and the classroom lectures and handouts prepared by teachers. Thus, it becomes necessary to go beyond the strictly linguistic features of mathematics language to consider possible difficulties relating to extralinguistic contexts of utterance, i.e., the places, persons, and times involved as well as the beliefs, intentions, presuppositions, and background knowledge of participants in mathematical discourse. Below, we isolate a few of the epistemological problems noted in our research, saving discussion of the co-occurence of syntactic, semantic, and pragmatic features in math discourse for our analysis of protocols in the next section.

Students who lack certain kinds of experience, or whose experience has been different from or even contradictory to the experiences presupposed by certain word problems, are apt to encounter difficulties. For example, students who are unfamiliar with an economic system that encourages competition and private enterprise are likely to misunderstand such business concepts as discounts, markups, wholesale, and retail. Or, if students happen to know only the tax rate in their locality, then it is not surprising that they suppose that rate applies everywhere, even if it is the unknown in the problem. Some students balk at the notion of solving for the tax rate because, in real life, the tax rate is always known and it is the amount of tax that needs to be calculated. However, because actual tax tables employ a rounding-off system that differs from the rule of thumb adopted by math texts, i.e., that we round off at .5 and above, there is the possibility of error even where the student has been encouraged to go into the marketplace to test his or her math skills.

The pedagogical advantages of incorporating real situations in an interactive framework, a common practice in most English as a second language (ESL) curricula, seem to be absent in the traditional mathematics curriculum. Communicative breakdowns are apt to occur when math texts and classroom lectures proceed in a rigid, lockstep manner, i.e., in a manner whereby the text or the teacher provides a statement plus explanation of a rule or property, demonstrates a few examples, and then gives the student problems to solve. It is often the case that these problems are unrelated to student experience, a problem noted above, which means that students have never had the opportunity to discuss the concepts that are involved.

A LINGUISTIC ANALYSIS OF SOME PROBLEM-SOLVING PROTOCOLS

In the previous section, we provided a linguistic model for categorizing the syntactic, semantic, and pragmatic features of the mathematics register. In this section, we focus on the interplay between these features that occurs when students are engaged in problem-solving tasks. Such a focus should make it clear that the three types of features are related in complex ways that make mastery of

the mathematics register difficult and that necessitate the special kinds of teaching materials and techniques discussed in the next section.

Each of the two problems cited here was presented to groups of students in beginning algebra classes. The sessions, led by a researcher-facilitator, were taped, transcribed, and analyzed to identify linguistic difficulties. Most of the students were non-native speakers of English with varying degrees of English proficiency. A few of the students were, however, native English speakers. For both of the problems, portions of three different transcripts are used to illustrate a variety of syntactic (SYN), semantic (SEM), and pragmatic (PRAG) features that were problematic for the participating students.

Problem 1: The sales tax is $15 on the purchase of a diamond ring for $500. What is the sales tax?

(S1 = first student; S2 = second student; R = researcher)

Problem 1
Transcript 1

SEM S1: That makes me confused sometimes in understanding *on the purchase of a diamond.*
R: Oh, so this phrase, *on the purchase of a diamond . . . ?*
S1: Right.
R: . . . confused you. Was it the term *on* that . . .
PRAG S1: Yeah, OK. I was, in my language what you sometimes have to do that, suppose if you purchase, like the purchase of 500, 500 dollars and sometimes we do like that way. It makes an understanding problem.
SEM S1: . . . now I'm saying that the 15 *on* the 500 or 15 dollars included in the 500 like 485 plus 15 dollars is 500. My purchase is 485 so I would like to say that the prepositional phrase *on the,* like suppose a customer bought 500 dollars goods and he paid 15 dollars tax on that and what percent sales tax did he pay on 500 dollars. Like that way, you know?

Problem 1
Transcript 2

PRAG S1: Well, we know here in Miami it's 5%. So you have to divide by . . .
SEM OK! 15 over 100, I mean 500. I don't know.
SYN S2: Can I help? I suggest that you divide 500 by 15 and that will give you the rate.

S1: Right!
R: Tell me again. You divide 500 by...
S1: 15.
R: Let's do it and see what we get.
(S1 calculates the answer)
S1: OK. It's 3%.

Problem 1
Transcript 3

SEM S2: The 15 dollars is the sales tax and the price of the ring is 500, so it would be 515 dollars. But now how do I get the sales tax *rate?* What do I have to do? Divide?
R: Keep going.
PRAG S2: What I'm thinking is . . . but then again, maybe it isn't plus 5%.

Discussion of Problem 1. Excerpts from these three problem-solving sessions involving this same sales tax problem reveal student difficulties on the syntactic, semantic, and pragmatic levels. In Transcript 1, the student admits confusion regarding the phrase *on the purchase of a diamond ring for $500.* This confusion might relate to the lexical meaning of the idiom *on the purchase of,* or to the precise reference of *the purchase,* which can be $500 or $485. The student chooses to disambiguate the problem by opting for an interpretation based on experience with his native language (Bengali). That is, in his country, (Bangladesh) speaking his language, the total purchase would be $500 minus $15 rather than $500 plus $15. Thus, this student exhibits difficulties related both to semantic (lexical) and pragmatic (epistemological) features.

In Transcript 2, the attempt to solve the problem reveals that S1 does not know how to translate the term *divide by* into math notation, i.e., doesn't know if 15 is the numerator or denominator. S2 makes a syntactic error in reversing the order of the numerator and the denominator (''500 by 15'' instead of ''15 by 500''). Yet, S1 gets the correct answer, meaning that the correct order was chosen despite the mistaken suggestion of S2. One wonders what S1 would have done if S2's suggestion had been correct!

In Transcript 2, S1 makes reference to the sales tax in Miami, as does S2 in Transcript 3. In the first case, S1 seems to think that this bit of knowledge will be helpful in setting up the solution, an idea that is quickly discarded. In the second case, S2, after exhibiting some semantic confusion regarding a perceived difference between sales tax and sales tax rate, is unable to abandon the notion that the answer is 5% despite the fact that dividing 15 by 500 does not support such a solution.

Problem 2: Two cars that are 375 miles apart and whose speeds differ by 5 miles per hour are moving toward each other. They will meet in 3 hours. What is the speed of each car?

Problem 2
Transcript 1
PRAG S1: The wording is what's difficult.
R: What's bad about the wording?
PRAG S1: That one with the cars. Why can't they just say ''One car . . . was driving x and the other car was driving x + 5?''
R: You want them to set it up for you, huh? (Laughter)
S1: Right!

Problem 2
Transcript 2
R: Rick, what are you doing over there. Help us out!
PRAG S1: OK. Well, for example, you have 2 cars. So I drew 2 cars (like blocks). And they're 375 miles apart, so I put 375 in the middle. ''Whose speeds differ by 5 miles per hour.'' They're moving *toward each other.* They will meet in 3 hours. What is the speed of each car?'' So what I'm beginning to do is I divided 375 by 2, so there's approximately 187 miles to the middle. Here's where I'm stuck. I am now trying to think up a formula. Maybe if I divide . . . 2 cars . . .

Problem 2
Transcript 3 (native English speakers)
S2: OK. Think out loud. Put the problem into context. 375 miles, um, whose speeds differ by 5 miles, so 50 or 55, 60 or 65. That's how they're going. One's going 5 miles an hour faster than the other, OK, let's assign a value to each car. Or for their speed. OK, so you got . . .
S1: x and $x + 5$.

Discussion of Problem 2. Problem 2 represents a type of problem widely reported to cause difficulties regardless of language proficiency. In Transcript 1, S1 complains about the wording, i.e., the textual difficulties involved in translating English into math notation, particularly in problems that do not relate to any obviously relevant real-life situation. In Transcript 2, S1 attempts to provide some reality by drawing a picture, but this only leads to an approximation, one that fails to come to grips with the relationship between distance, rate, and time necessary to solve the problem.

In Transcript 3, S1 and S2 are both native speakers of English who were instructed to follow a heuristic procedure similar to the four-step method promoted by Polya (1945). In this case, the attempt to put the problem into a context leads to an appeal to a piece of experience that enables them to approximate both the solution equation (x and $x + 5$) and the solution (60 or 65). Again, however, as in Transcript 2, they do not see any relationship between these numbers and variables and the distance, rate, time formula needed to set up the solution. In fact, a non-native English speaker who joined the discussion later did bring this important piece of mathematical knowledge with him, and this enabled the three students to work backwards from 60 and 65, x and $x + 5$, to the appropriate solution equation.

APPLICATIONS: A LANGUAGE APPROACH TO TEACHING MATHEMATICS

The previous examples illustrate the difficulties students face in attempting to acquire the mathematics register. They also suggest an approach to help students acquire this register: the development and implementation of interactive exercises and related teaching techniques that require students to discuss problems en route to solutions. The very act of talking through problems—of discussing various strategies for beginning the problem (or "attacking" it) and using the language to solve the problem—enable students to gradually become comfortable listening to and using mathematics language.

Although problem-solving sessions with students were initially used for data gathering, they also provided direction for an instructional approach. If students were given an opportunity to work together in a systematic way to develop competence in both listening and using math language, followed by practice in the written language, they could move beyond the stage of considering math language as a barrier, to seeing it as a tool for doing mathematics. That is, the language became an instrument that students could actively use, sometimes translating it into more familiar language terms. Moreover, they could benefit from mutual discussion and experience.

However, the process needed to be systematized to guide the practice and the acquisition. Thus, materials have been developed that provide students with an opportunity to understand math language in context and then to practice using it, both in explaining and actually solving the problems. Although the materials are intended to be used in paired tutoring sessions (and thus, are laid out with the tutor's page on the left and the comparable student's page on the right), they have also been used as classroom texts and as the basis for homework assignments. If they are used by students individually, the student is asked to look at the tutor's section for directions and verification. (An audio cassette of the tutor's role is being prepared for individual use with one of the units.)

The five units in these materials, *English Language Skills for Basic Algebra* (see Crandall et al., 1987), provide a review of basic mathematics and algebraic expressions, an introduction to equations and inequalities, an approach to and practice in solving a variety of word problems, an introduction to properties and theorems, and a glossary of key terms, symbols, types of numbers, and common phrases used in word problems. (The glossary is intended as a reference to be used throughout the program.) These units are cross-referenced to five college algebra textbooks used in the United States. The materials are carefully designed to help students move from an understanding of numerical and algebraic expressions, to the use of these in solving equations, inequalities, and word problems, and being able to understand definitions and theorems. For example, students might begin by being asked to understand that when their partners say "sixteen increased by eleven" that the appropriate numerical expression is $16 + 11$, just as it is if their partners say "sixteen plus eleven," "eleven added to sixteen," or "eleven more than sixteen." A subsequent exercise might introduce *square root* or an equation in which this same expression is required. As students progress through the materials, sometimes acting as tutors and other times as students, they become increasingly able to use math language to think about, discuss, and set up and solve problems. The unit on solving word problems also introduces a number of strategies for coping with the major difficulties in word problems: for example, finding ways of focusing on what is known (or unknown) and what is asked for, using strategies such as drawing diagrams to assist in the process; and watching out for key words such as *toward* (in mileage/distance/rate problems), *ago* or *will be* in age problems, and so forth.

Verbalization exercises can serve as a major component of class instruction or as a kind of supplementary tutorial or lab work. The materials have been used with the teacher and entire class in a traditional manner, in learning centers with two students working together, or in paired practice in class. They have been used in ESL classes as a means of increasing the academic content of English language classes and in algebra classes, as the basis for homework, small-group work, or even full class discussion.

A language approach to the teaching of mathematics provides multiple opportunities for students to develop listening, speaking, reading, and writing skills as they are acquiring mathematical skills. There are a number of other ways in which students can be assisted in developing language skills appropriate to their mathematics courses. For example, although the typical lesson in an algebra class focuses almost exclusively on the presentation of a problem type with sample demonstrations, followed by student practice, it is possible for teachers to provide more transition into problem solving by explaining the various linguistic items, giving students an opportunity to work together to solve problems (while the teacher listens for problematic language and math features), and asking discussion questions. These activities include:

- providing opportunities for students to "create" word problems, using numbers and other information provided by the teacher;
- preparing students for and then asking essay-type questions on exams, which force students to use math language rather than mechanically calculate solutions to problems;
- using dialogue-journals with students, providing students with an opportunity to write about their math problems, in a written dialogue between teacher and student about the nature of mathematics and the individual student's progress through the course. These journals also serve as a place to answer questions that the student may not have felt comfortable articulating in front of the entire class.

In summary, a language approach to mathematics education offers students the opportunity to acquire competence in both understanding and using mathematics language: in being able to understand instructor's presentations, text explanations, problems, and tests; in being able to ask questions and discuss problems in class; and in being able to work through algebraic, geometric, or other problems, using the mathematics language appropriate to the task. Math-anxious students who seem unable to respond when confronted by the mathematics register can be assisted in learning to cope with the language and even to effectively use that language in doing mathematics. However, this can only be accomplished when there is a conscious effort on the part of the instructor to provide activities that will facilitate the learning of this complex and difficult register. To expect students to "pick up" the language as they read the texts and listen to explanations in class is to court the kind of failure that too many students, both language minority and English-speaking, meet in their mathematics courses. The direction of our work, as applied linguists, and the work of some of our colleagues in the field of mathematics education, is to provide materials that will enable instructors to assist students in both the language and principles of mathematics.

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REFERENCES

Aiken, L. R. (1971). Verbal factors and mathematics learning: A review of research. *Journal for Research in Mathematics Education, 2*(4), 304–312.

Burns, M., Gerace, W. J., Mestre, J. P., & Robinson, H. (1983). The current status of Hispanic

technical professionals: How can we improve recruitment and retention? *Integrated Education, 20,* 49–55.

Carnap, R. (1955). Foundations of logic and mathematics. *International Encyclopedia of Unified Science, 1*(3), 139–212. Chicago, IL: University of Chicago Press.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 13,* 16–30.

Cossio, M. G. (1978). The effects of language on mathematics placement scores in metropolitan colleges. *Dissertation Abstracts International, 38,* 4002A–4003A (University Microfilms No. 77–27,882)

Crandall, J., Dale, T. C., Rhodes, N. C., & Spanos, G. (1987). *English language skills for basic algebra.* Englewood Cliffs, NJ: Prentice-Hall.

Crandall, J., Dale, T. C., Rhodes, N. C., & Spanos, G. (in press). The language of mathematics: The English barrier. *Proceedings of the 1985 Delaware Symposium on Language Studies VII.* Newark, DE: University of Delaware Press.

Cuevas, G. (1984). Mathematical learning in English as a second language. *Journal for Research in Mathematics Education, 15,* 134–144.

Cummins, J. (1981). The role of primary language development in promoting educational success for language minority students. In California State Department of Education, Office of Bilingual Bicultural Education (Ed.), *Schooling and language minority students* (pp. 3–50). Los Angeles: Evaluation, Dissemination and Assessment Center, California State University.

Dawe, L. (1984, August). *A theoretical framework for the study of the effects of bilingualism on mathematics teaching and learning.* Paper presented at the Fifth International Congress on Mathematical Education, Adelaide, Australia.

DeAvila, E., & Havassy, B. (1974). *IQ tests and minority children.* Austin, TX: Dissemination Center for Bilingual Bicultural Education.

Dolciani, M. P., & Wooton, W. (1970). *Book one, modern algebra, revised edition.* Boston: Houghton Mifflin.

Duran, R. (1979). *Logical reasoning skills of Puerto Rican bilinguals* (Final Report Grant No. NIE-G-78—135). Washington, DC: National Institute of Education.

Firsching, J. T. (1985). *Conversational implicatures: Mathematics word problems and teacher explanations* (Vol. 1). Unpublished dissertation, Georgetown University, Washington, DC.

Gerace, W. J., & Mestre, J. P. (1983). *A study of the algebra acquisition of Hispanic and Anglo ninth graders* (Final Report on NIE Contract 400-81-0027). Amherst, MA: University of Massachusetts, Department of Physics and Astronomy. (ERIC ED 231613)

Halliday, M. A. K. (1975). Some aspects of sociolinguistics. In E. Jacobson (Ed.), *Interactions between linguistics and mathematics education* (pp. 64–73). *Final Report of the Symposium Sponsored by UNESCO, CEDO and ICMI,* Nairobi, Kenya, September 1–11, 1974. (UNESCO Report No. ED-74/CONF. 808. Paris: UNESCO, 64–73)

Kessler, C., Quinn, M. E., & Hayes, C. W. (1986). Processing mathematics in a second language: Problems for LEP children. To appear in *Proceedings of the 1985 Delaware Symposium on Language Studies VII.* Newark, DE: University of Delaware Press.

Knight, L., & Hargis, C. (1977). Math language ability: Its relationship to reading in math. *Language Arts, 54,* 423–428.

Mestre, J. P. (1981). Predicting academic achievement among bilingual Hispanic college technical students. *Educational and Psychological Measurement, 41,* 1255–1264.

Mestre, J. P. (1986). Teaching problem-solving strategies to bilingual students: What do research results tell us? *International Journal of Mathematical Education in Science and Technology, 17,* 393–401.

Mestre, J. P., Gerace, W. J., & Lochhead, J. (1982). The interdependence of language and translational math skills among bilingual Hispanic engineering students. *Journal of Research in Science Teaching, 19,* 399–410.

Moreno, S. (1970). Problems related to present testing instruments. *El Grito, 3,* 25–29.
Morris, C. W. (1955). Foundations of the theory of signs. *International Encyclopedia of Unified Science. 1*(2) 78–137. Chicago, IL: University of Chicago Press.
Munro, J. (1979). Language abilities and math performance. *Reading Teacher, 32,* 900–915.
Polya, G. (1945). *How to solve it.* Princeton, NJ: Princeton University Press.
Ramirez, M., & Gonzalez, A. (1972). Mexican Americans and intelligence training. In M. M. Mangold (Ed.), *La causa Chicana: The movement for justice* (pp. 137–147). New York: Family Services Association of America.
Thorndike, E. L. (1912). The measurement of educational products. *School Review,* 20, 289–299.

Chapter 14

Bilinguals' Logical Reasoning Aptitude: *A Construct Validity Study*

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BILINGUALISM AND COGNITION

How does the proficiency in a language affect the ability to solve problems? This is an important question for educators, testing and assessment specialists, and cognitive psychologists. Educators are interested in better understanding ways in which non-English background students can function effectively in classrooms offering instruction in one language or another. They need to understand whether the choice of a language of learning and problem solving will affect display of the underlying cognitive skills of bilingual students. Educators are especially interested in understanding how students' earlier schooling experiences and backgrounds are related to their language proficiency skills, as well as to the cognitive skills in which students have developed competence.

Testing specialists in the schools have related concerns. They are often called upon by teachers and school staff to answer questions about how to best serve bilingual students; it is often the burden of testing specialists to assess the cognitive and language proficiency skills of bilingual students. Further, when trained appropriately, testing specialists can also help contribute to explanations of how bilingual students' backgrounds might be related to their learning ability in either language.

Cognitive psychologists are more removed from the everyday contexts of schools. However, they too are interested in some of these questions. They are interested in developing scientific theories of how the mind functions when operating in one language or another. Like educators, cognitive psychologists