Problem 5. Constant returns to saving.

Consider the following problem.

$$\max_{c_0, k_1, c_1, k_2, \ldots} \sum_{t=0}^{\infty} \frac{\beta^t}{1-\alpha} c_t^{1-\alpha} \quad \text{s.t.} \quad k_{t+1} = R(k_t - c_t) \text{ for all } t \geq 0, \text{ and } k_0 \text{ given,}$$

$$\text{and} \quad \lim_{T\to\infty} R^{-T} k_T \geq 0$$

a. Show that the constraint $\lim_{T\to\infty} R^{-T} k_T \geq 0$ is equivalent to imposing a constraint that the present discounted value of lifetime consumption is less than or equal to initial wealth.

b. Give an example of a (non-feasible) time path of consumption and capital which violates the last constraint.

c. Write Bellman's equation for this problem, substituting in the law of motion for capital. (You can ignore the last constraint as you write Bellman's equation.)

d. Guess and verify a solution of the form $V(k) = Ak^{1-\alpha}$ for some constant A. If possible, calculate A.

d. State the policy functions in the forms $c_t = Bk_t$ and $k_{t+1} = Ck_t$, and calculate B and C, if possible.

f. Show that for the optimal policies from (e), the final constraint on the problem is *not* violated.

g. Explain the intuition behind the behavior of the model. Is there a steady state? What restriction on the exogenous parameters of the model would cause it to have a steady state?

Problem 5 – Constant Returns to Savings

Consider the problem,

$$\max_{c_0,k_1,c_1,k_2,\dots} \sum_{t=0}^{\infty} \left(\frac{\beta^t}{1-\alpha}\right) c_t^{1-\alpha} \qquad s.t.\quad k_{t+1} = R(k_t - c_t) \quad \forall\, t \geq 0,\ \ \&\, k_0\ is\ given.$$

$$and \quad s.t.\ \lim_{T\to\infty} R^{-T} k_T \geq 0$$

A) Show that the constraint $\lim_{T\to\infty} R^{-T} k_t \geq 0$ is equivalent to imposing a constraint that the present discounted value of lifetime consumption is less than or equal to initial wealth.

Assuming people are dead at $time = T$, and therefore cannot appreciate consumption then, the present discounted value of lifetime consumption is equivalent to,

$$\frac{c_0}{R^0} + \frac{c_1}{R^1} + \frac{c_2}{R^3} + \frac{c_4}{R^4} + \cdots \frac{c_{T-1}}{R^{T-1}}$$

Thus the constraint,

$$\left(\frac{c_0}{R^0} + \frac{c_1}{R^1} + \frac{c_2}{R^3} + \frac{c_4}{R^4} + \cdots \frac{c_{T-1}}{R^{T-1}}\right) \leq k_0 \quad , \quad where\ \ c_t = k_t - \frac{k_{t+1}}{R}$$

$$\Rightarrow \left(\frac{\left(k_0 - \frac{k_1}{R}\right)}{1} + \frac{\left(k_1 - \frac{k_2}{R}\right)}{R^1} + \frac{\left(k_2 - \frac{k_3}{R}\right)}{R^2} + \frac{\left(k_3 - \frac{k_4}{R}\right)}{R^3} + \cdots \frac{\left(k_{T-1} - \frac{k_T}{R}\right)}{R^{T-1}}\right) \leq k_0$$

$$\Rightarrow \left(\left(k_0 - \frac{k_1}{R}\right) + \left(\frac{k_1}{R} - \frac{k_2}{R^2}\right) + \left(\frac{k_2}{R^2} - \frac{k_3}{R^3}\right) + \left(\frac{k_3}{R^3} - \frac{k_4}{R^4}\right) + \cdots \frac{\left(k_{T-1} - \frac{k_T}{R}\right)}{R^{T-1}}\right) \leq k_0$$

You have hopefully detected a subtle pattern of alternating minus and plus signs before identical terms, simplifying,

$$\Rightarrow \quad k_0 - \frac{k_T}{R^T} \leq k_0$$

$$\Rightarrow \quad -\frac{k_T}{R^T} \leq 0$$

$$\Rightarrow \quad \frac{k_T}{R^T} \geq 0 \qquad\qquad not\ sure\ where\ the\ "\lim_{T\to\infty}"\ comes\ in \dots\ ,sorry$$

B) Give an example of a (non-feasible) time path of consumption and capital which violates the last constraint.

A consumption time path in which consumption increases continuously forever would violate this constraint.

C) Convert this problem into a Bellman's equation, substituting in the law of motion for capital. (You can ignore the last constraint as you write Bellman's equation).

The Bellman Equation for this problem, substituting in the law for motion for capital.

$$V(k) = \max_{c,k'} \left(\frac{1}{1-\alpha}\right) c^{1-\alpha} + \beta\, V(k') \qquad s.t.\ k' = R(k - c\,)$$

$$\Rightarrow \quad V(k) = \max_{c,k'} \left(\frac{1}{1-\alpha}\right) c^{1-\alpha} + \beta\, V\big(R(k - c\,)\big)$$

D Part I) Guess and verify a solution of the form $V(k) = Ak^{1-\alpha}$ for some constant A. If possible, calculate A.

$$V(k_t) = Ak_t^{\,1-\alpha} = \max_{c,k'}\left(\frac{1}{1-\alpha}\right)c^{1-\alpha} + \beta\, Ak'^{\,1-\alpha} \quad \textit{(note the}\quad \textit{'above the k term)}$$

$$k' = R(k-c) \quad \Rightarrow \quad c = k - \frac{k'}{R}$$

$$\Rightarrow Ak_t^{\,1-\alpha} = \max_{k'}\left(\frac{1}{1-\alpha}\right)\left(k - \frac{k'}{R}\right)^{1-\alpha} + \beta\, Ak'^{\,1-\alpha}$$

Now, solving a maximization problem like this is simple,

$$\left(\frac{\partial}{\partial k'}\right)\left[\left(\frac{1}{1-\alpha}\right)\left(k - \frac{k'}{R}\right)^{1-\alpha} + \beta\, Ak'^{\,1-\alpha}\right] = 0$$

$$\Rightarrow \left(\frac{1}{1-\alpha}\right)\left(k - \frac{k'}{R}\right)^{1-\alpha-1}(1-\alpha)\left(-\frac{1}{R}\right) + (1-\alpha)\beta Ak'^{\,1-\alpha-1} = 0$$

$$\Rightarrow \left(-\frac{1}{R}\right)\left(k - \frac{k'}{R}\right)^{-\alpha} + (1-\alpha)\beta Ak'^{-\alpha} = 0$$

$$\Rightarrow \quad (1-\alpha)\beta Ak'^{-\alpha} = \frac{1}{R}\left(k - \frac{k'}{R}\right)^{-\alpha}$$

I don't see any way to solve for A without it being in terms of both k & k'.

$$A = \frac{\frac{1}{R}\left(k - \frac{k'}{R}\right)^{-\alpha}}{(1-\alpha)\beta k'^{-\alpha}} = \frac{1}{(1-\alpha)R\beta}\left(\frac{k}{k'} - \frac{1}{R}\right)^{-\alpha}$$

D Part II) State the policy functions in the forms $c_t = Bk_t$ & $k_{t+1} = Ck_t$, and calculate B and C, if possible. **(note that with my notation, $k_t = k$ & $k_{t+1} = k'$)**

Back to,

$$(1-\alpha)\beta Ak'^{-\alpha} = \frac{1}{R}\left(k - \frac{k'}{R}\right)^{-\alpha}$$

Using MATLAB,

$$k' = k*R - R*e^{\left(\ln - \frac{R}{(\alpha-1)\beta A}\right)} + \frac{1 + R*e^{\frac{\ln\left(k*\frac{R}{\ln\left(-\frac{\frac{\frac{\frac{1}{\beta}}{A}}{R}}{-1+\alpha}\right)}\right)*\alpha)}{\alpha}}}{\alpha}$$

MATLAB isn't helpful here – and at least it's k'equals something in terms of k - but it is very messy.

c works out to be even bigger. I am thus either doing something wrong, of this is a very messy problem. I'm inclined to think the later.

Now let's try to solve this via Euler's Method.

$$V_t(k_t) = \frac{1}{1-\alpha} c_t^{1-\alpha} + \beta V_{t+1}(k_{t+1}) + \lambda_t(R(k_t - c_t) - k_{t+1})$$

First Order Conditions

$$(1)\ \frac{\partial \mathcal{L}}{\partial c_t}: \qquad c_t^{-\alpha} - R\lambda_t = 0$$

$$(2)\ \frac{\partial \mathcal{L}}{\partial k_{t+1}}: \qquad \beta V'_{t+1}(k_{t+1}) - \lambda_t = 0$$

$$(3)\ \frac{\partial \mathcal{L}}{\partial \lambda_t}: \qquad R(k_t - c_t) - k_{t+1} = 0$$

Envelope Condition

$$(4)\ V'_t(k_t) = R\lambda_t \qquad = \frac{\partial \mathcal{L}}{\partial k_t}$$

So – the problem asks for a solution in the form of $c_t = Bk_t$ & $k_{t+1} = Ck_t$, but I don't see how that is possible – there is no way to eliminate c_t from the $k_{t+1} = (\dots)$ equation – at least that I can see anyway. Going forward I'm going to assume that the question asks to find the steady state and eigenvalues and vectors for this problem.

$$(4) \implies V'_{t+1}(k_{t+1}) = R\lambda_{t+1}$$

$$(4)\&\ (2) \implies \frac{\lambda_t}{\beta} = R\lambda_{t+1} \qquad \implies \qquad \lambda_t = \beta R \lambda_{t+1}$$

$$\textit{Combining with } (1) \qquad \lambda_t = \frac{c_t^{-\alpha}}{R}$$

$$\implies \qquad \boldsymbol{\frac{c_t^{-\alpha}}{R} = \beta c_{t+1}^{-\alpha}}$$

$$(3) \qquad \boldsymbol{R(k_t - c_t) = k_{t+1}}$$

Rearranging a bit,

$$\frac{c_t^{-\alpha}}{R} = \beta c_{t+1}^{-\alpha}$$

$$\implies \quad c_t^{-\alpha} = (\beta R) c_{t+1}^{-\alpha}$$

$$\implies \quad c_t = (\beta R)^{-\frac{1}{\alpha}} c_{t+1}$$

We get the following system of difference equations.

$$c_{t+1} = (\beta R)^{\frac{1}{\alpha}} c_t$$

$$k_{t+1} = R(k_t - c_t)$$

Steady State – a steady state is a value c such that $c_t = c_{t+1} = c^* \;\; \forall t$

$$0 = (\beta R)^{\frac{1}{\alpha}} c^* \qquad ; \qquad c^* = 0$$
$$0 = R(k^* - c^*) \qquad ; \qquad c^* = k^*$$

There will be one steady state at (0,0). This is typically the case with economic models (you know, with not capital and not consumption, you'll stay that way forever) and is uninteresting.

There is another set of steady states at the line $c = k$

Rewriting our two conditions that describe the optimal allocation.

$$\begin{pmatrix} c_{t+1} \\ k_{t+1} \end{pmatrix} = \begin{pmatrix} (\beta R)^{\frac{1}{\alpha}} & 0 \\ -R & R \end{pmatrix} \begin{pmatrix} c_t \\ k_t \end{pmatrix}$$

Finding EigenVectors & EigenValues

This is a simple 2×2 matrix.

$$Given\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$Trace:\ T = a + d$$

$$Determinant: D = ad - cb$$

$$EigenValues: \quad \begin{aligned} \mathcal{L}_1 &= \frac{T}{2} + \left(\frac{T^2}{4} - D\right)^{\frac{1}{2}} \\ \mathcal{L}_2 &= \frac{T}{2} - \left(\frac{T^2}{4} - D\right)^{\frac{1}{2}} \end{aligned}$$

Our First Eigenvalue

$$\mathcal{L}_1 = \frac{T}{2} + \left(\frac{T^2}{4} - D\right)^{\frac{1}{2}} = \frac{\left((\beta R)^{\frac{1}{\alpha}} + R\right)}{2} + \left[\frac{\left((\beta R)^{\frac{1}{\alpha}} + R\right)^2}{4} - \left((\beta R)^{\frac{1}{\alpha}} R\right)\right]^{\frac{1}{2}}$$

According to Wolphramalpha: bit.ly/bKdOOZ

$\boldsymbol{\mathcal{L}_1 = R}$, *assuming that* $\beta, R \;\&\; \alpha > 0,$ *which we can assume*

& *we assume* $(\beta R)^{\frac{1}{\alpha}} \leq R,$ *which we can assume with normal ranges for* $\beta, R \;\&\; \alpha$

Showing that $(\beta R)^{\frac{1}{\alpha}} \leq R \implies$ this implies that $R \geq 1 \;\&\; \approx 1,$ and $0 < \beta \leq 1,$ and $0 < \alpha \leq 1$

Showing that this is a reasonable assumption considering this problem, here is a spreadsheet that goes over a number of examples to show how this condition likely holds. **bit.ly/adsdWE**

R is rate of return. β is our discounting. α is the "c_t" share of utility/production (see Cobb-Douglas production function)

Our Second Eigenvalue

$$\mathcal{L}_2 = \frac{T}{2} - \left(\frac{T^2}{4} - D\right)^{\frac{1}{2}} = \frac{\left((\beta R)^{\frac{1}{\alpha}} + R\right)}{2} - \left[\frac{\left((\beta R)^{\frac{1}{\alpha}} + R\right)^2}{4} - \left((\beta R)^{\frac{1}{\alpha}} R\right)\right]^{\frac{1}{2}}$$

According to Wolphframalpha; **bit.ly/dsYp5S**

$\boldsymbol{\mathcal{L}_2} = (\boldsymbol{\beta R})^{\frac{1}{\alpha}},\quad if\;\; \beta, \alpha, R > 0 \quad \& \quad > R$

The eigenvalues are R and $(\beta R)^{\frac{1}{\alpha}}$

Eigenvectors:

$$\begin{pmatrix} (\beta R)^{\frac{1}{\alpha}} & 0 \\ -R & R \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

With the b cell zero this becomes easy;

For the Eigenvalue $\mathcal{L}_1 = R$

$$\begin{pmatrix} \mathcal{L}_1 - 'd' \\ 'c' \end{pmatrix} = \begin{pmatrix} R - R \\ -R \end{pmatrix} = \begin{pmatrix} 0 \\ -R \end{pmatrix} \equiv \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$$

For the Eigenvalue $\mathcal{L}_2 = (\beta R)^{\frac{1}{\alpha}}$

$$\begin{pmatrix} \mathcal{L}_2 - 'd' \\ 'c' \end{pmatrix} = \begin{pmatrix} (\beta R)^{\frac{1}{\alpha}} - R \\ -R \end{pmatrix}$$

Once again, we assume that $(\beta R)^{\frac{1}{\alpha}} \leq R \implies$ the top term is a small negative number,

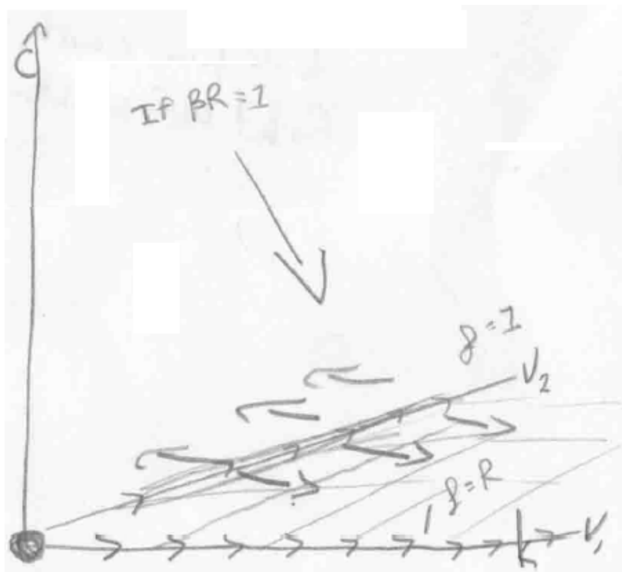
$$= \begin{pmatrix} \textbf{\textit{Small Positive \#}} \\ \mathbf{1} \end{pmatrix}$$

Dynamics

We know that $R \geq 1$ – it's the rate of return.

$(\beta R)^{\frac{1}{\alpha}}$ however could be <1 or >1, giving different dynamics.

If $BR = 1$, you might see dynamics like the following,



Problem 6. Optimal saving under exponential utility.

Consider an individual who maximizes exponential utility over an infinite horizon:

$$\max_{\{c_t, W_{t+1}\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \left(\frac{-1}{\alpha}\right) e^{-\alpha c_t} \quad \text{s.t. } W_{t+1} \ = \ R(W_t - c_t) + y_{t+1}$$

$$\text{and s.t. } \lim_{t\to\infty} \frac{W_t}{R^t} \ \geq \ 0\ ; \quad \text{and } W_0 \text{ given.}$$

Here $\beta \in (0,1)$ and $\alpha > 0$. W represents current wealth; c is consumption; y is income, which we will assume is constant: $y_t = y > 0$ for all t. $R > 1$ is one plus the interest rate.

a. Show that imposing the last constraint on the worker's problem is equivalent to imposing the following constraint:

$$\sum_{t=0}^{\infty} \frac{c_t}{R^t} \ \leq \ \sum_{t=1}^{\infty} \frac{y_t}{R^t} \ + \ W_0$$

b. Write a Bellman equation describing the individual's maximization problem, writing the value function in terms of current resources W_t. (Note: you may ignore the last constraint when writing the Bellman equation.)

c. *Guess* that the value function has the form $V(W_t) = -Ae^{-ZW_t}$ where $A > 0$ and $Z > 0$ are unknown constants. Rewrite the Bellman equation in terms of this functional form.

d. Find the policy functions for c_t and W_{t+1} in terms of W_t and in terms of the unknown constants A and Z.

d. Show that the guess is correct. Find the constant Z. (You are not required to find A.)

e. Prove that under the optimal policy, $c_t = \frac{R-1}{R} W_t + constant$. Describe the time path of W_t under the optimal policy. Does this optimal policy satisfy the last constraint on the problem?

A)

$$\max_{\{c_t, W_{t+1}\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \left(-\frac{1}{\alpha}\right) e^{-\alpha c_t} \; s.t. \; W_{t+1} = R(W_t - c_t) + y_{t+1}$$

$$and \; s.t. \; lim_{t\to\infty} \frac{W_t}{R^t} \geq 0$$

W_0 given
$\beta \epsilon (0,1) \alpha > 0$
$W = current\ wealth$
$c = consumption$
$y = income, a\ constant$
$y_t = y > 0\ \forall f$
$R > 1, = 1 + r$

a) Show that:

$$\lim_{t\to\infty} \frac{W_t}{R^t} \geq 0 \text{, is equivolent to } \sum_{t=0}^{\infty} \frac{C_t}{R^t} \leq \sum_{t=1}^{\infty} \frac{Y_t}{R^t} + W_0$$

$$\lim_{t\to\infty} \frac{W_t}{(1+r)^t} \geq 0 \rightarrow \frac{W_t}{1} \geq 0$$

$(1+r)^t$ goes to 1

$$\frac{c_t}{1} \leq \frac{y_t}{1} + W_0$$

$\Rightarrow W_0 + y_t + c_t \geq 0 \;:\; and\ with\ W_t = W_0 + y_t + c_t$

$\Rightarrow W_t \geq 0$

This says that a person cannot borrow – debt can never be positive (imagine what strange/complex dynamics might ensure if that were possible, even in a simple utility max model like this). It is the equivalent to saying that the sum of one's consumption must be less than or equal to the sum of their income and initial wealth.

B)

$W: state\ variable$

$r\ is\ a\ constaint$

$y\ is\ a\ constant\ (if\ there\ were\ shocks - econ205b - it\ could\ turn\ into\ a\ state\ variable)$

$$V(W_t) = \max_{s_t, W_{t+1}} -\frac{1}{\alpha} e^{\alpha c} + \beta\, V(W_{t+1}) \qquad s.t.\ W_{t+1} = R(W_t - c_t) + y$$

We guess – Ae^{-zW_t}

$$-Ae^{-zW} = \max_{c, W'} -\frac{1}{\alpha} e^{\alpha(c)} - A\beta e^{-zW'}$$

$$W' = R(W - c) + y$$

$$c = W + \frac{y - W'}{R}$$

.

Plugging in c

$$\Rightarrow -Ae^{-zW} = \max_{W'} -\frac{1}{\alpha}e^{\alpha\left(W+\frac{y-W'}{R}\right)} - A\beta e^{-zW'}$$

D) (D part I)

Solving the Maximization problem

$$\left(\frac{\partial}{\partial W'}\right)(-Ae^{-zW}) = \left(\frac{\partial}{\partial W'}\right)\left[-\frac{1}{\alpha}e^{\alpha\left(W+\frac{y}{R}-\frac{W'}{R}\right)} - A\beta e^{-zW'}\right] = 0$$

$$\Rightarrow -\left(\frac{1}{R}\right)e^{-\alpha\left(W+\frac{y}{R}-\frac{W'}{R}\right)} + ZA\beta e^{-zW'} = 0$$

$$\Rightarrow \quad e^{-\alpha\left(W+\frac{y}{R}-\frac{W'}{R}\right)} = RZA\beta e^{-zW'}$$

Now linearize this

$$\alpha\left(W + \frac{y}{R} - \frac{W'}{R}\right) = \ln(ZA\beta R) - ZW'$$

Keep in mind that W is a constant here, along with $y, R, Z, \& \ A$.

Solving for W' (our first policy function)

$$W'(W) = \left(\frac{R\alpha}{\alpha + RZ}\right)W + \widetilde{W_0} \quad , \quad where\ \widetilde{W_0} = \left(\frac{R}{\alpha + RZ}\right)\ln(ZA\beta R)$$

Now solving for c (our second policy function)

$$c = W + \frac{y - W'}{R} \quad , \quad with\ W'\ solved\ for\ above$$

$$= W + \frac{y - \left(\frac{R\alpha}{\alpha + RZ}W + \widetilde{W_0}\right)}{R} = W + \frac{y}{R} - \frac{\left(\frac{R\alpha}{\alpha + RZ}W + \widetilde{W_0}\right)}{R}$$

$$= \left(\frac{R(\alpha + RZ)}{R(\alpha + RZ)}\right)W - \frac{R\alpha}{R(\alpha + RZ)}W + \left[\frac{y}{R} - \frac{\widetilde{W_0}}{R}\right]$$

$$= \left(\frac{R^2 Z}{R(\alpha + RZ)}\right)W + C_0 \ , \ where\ \ C_0 = \left[\frac{y}{R} - \frac{\widetilde{W_0}}{R}\right]$$

$$= \left(\frac{RZ}{\alpha + RZ}\right)W + C_0 \ , \ where\ \ C_0 = \left[\frac{y}{R} - \frac{\widetilde{W_0}}{R}\right]$$

D) (D part II)

Was the guess correct? - To find out we need to plug in our guess.

If it is correct, $-Ae^{-ZW} = Our\ Policy\ Function\ Plugged\ Into\ Our\ Objective\ Function.$

$$-Ae^{-ZW} = -\frac{1}{\alpha}e^{\alpha(c)} - A\beta e^{-zW'}$$

$$-Ae^{-ZW} = -\frac{1}{\alpha}e^{\alpha\left(\left(\frac{RZ}{\alpha+RZ}\right)W+C_0\right)} - A\beta e^{-z\left(\left(\frac{R\alpha}{\alpha+RZ}\right)W+\widetilde{W_0}\right)}$$

Note that

$$Z = \alpha\left(\frac{RZ}{\alpha + RZ}\right)$$
$$\Rightarrow (\alpha + RZ)Z = \alpha RZ$$
$$\Rightarrow \alpha + RZ = \alpha R$$
$$\Rightarrow Z = \frac{aR - a}{R}$$
$$\Rightarrow \boldsymbol{Z = \alpha\left(\frac{R-1}{R}\right)}$$

Setting up to solve for A

$$A = -\frac{1}{\alpha}e^{-\alpha C_0} + A\beta e^{-ZW_0}$$

Solving for A with our known value for Z would be a pain – but definitely doable.

Part e

Show that the Optimal Policy is

$$c_t = \frac{R-1}{R}W_t + constant$$

Remember that

$$c = \left(\frac{RZ}{\alpha + RZ}\right)W + C_0$$

With $Z = \alpha\left(\frac{R-1}{R}\right)$

$$\Rightarrow RZ = \alpha(R-1)$$

$$c = \left(\frac{\alpha(R-1)}{\alpha + \alpha(R-1)}\right)W + C_0$$
$$= \left(\frac{(R-1)}{1 + (R-1)}\right)W + C_0$$
$$= \left(\frac{R-1}{R}\right)W + C_0\ , \qquad shown!$$

Problem 7. Climate change.

Consider a farmer who chooses consumption and saving over time, who is subject to random changes in climate which affect her productivity. Specifically, if productivity today is $A > 0$, then next period productivity will be AG with probability one half, and A/G with probability one half, for some number $G > 1$. Her maximization problem is:

$$\max_{c_0, k_1, c_1, k_2, \ldots} \quad E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t \quad \text{subject to:} \quad c_t + k_{t+1} = A_t k_t^\alpha \text{ for all } t \geq 0$$

$$\text{and} \quad A_{t+1} = A_t G \text{ with prob. } \frac{1}{2}, \quad \frac{A_t}{G} \text{ with prob. } \frac{1}{2}$$

$$\text{and } k_0,\ A_0 \text{ given.}$$

The notation E_0 before the sum of utilities here represents the "expectation" of utility.

a. How many state variables are there? Are they exogenous or endogenous?

b. Write Bellman's equation for this problem. Be sure to take into account the two possible future values of A.

c. Let's *guess* that the value function takes the form $C \ln A + D \ln k + E$, where C, D, and E are unknown constants. Rewrite the Bellman equation using this functional form.

d. Solve the maximization problem to find the policy functions in terms of the unknown parameters.

e. Substitute the policy functions into the value function and check whether your guess is correct.

f. Solve for C and D (you may also solve for E if you are really bored and have nothing else to do with your time.)

g. Considering the law of motion for A and the policy functions, describe the behavior of the model as well as you can.

This is the Robinson Crusoe problem with shocks to productivity.

State variables k_t & A_t

The Bellman Equation.

$$V(k, A) = \max_{c,k'} \ln(c) + \frac{1}{2}\beta\left[V(k', AG) + V\left(k', \frac{A}{G}\right)\right] \qquad s.t.\ \ k' = Ak^\alpha - c$$

Now guess that $V(k, A)$ takes the form $C\ln(A) + D\ln(k) + E$

This implies,

$$
\begin{aligned}
V(k,A) &= C\ln(A) + D\ln(k) + E \\
&= \max_{k'} \ln\left(Ak^{\alpha} - k'\right) + \frac{1}{2}\beta\left[\left[C\ln(AG) + D\ln(k') + E\right] + \left[C\ln\left(\frac{A}{G}\right) + D\ln(k') + E\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow C\ln(A) + D\ln(k) + E & \\
&= \max_{k'} \ln\left(Ak^{\alpha} - k'\right) \\
&+ \frac{\beta}{2}\left[\left[C\ln(A) + \cancel{C\ln(G)} + D\ln(k') + E\right] + \left[C\ln(A) - \cancel{C\ln(G)} + D\ln(k') + E\right]\right] = E
\end{aligned}
$$

$$
\Rightarrow C\ln(A) + D\ln(k) + E = \max_{k'} \ln\left(Ak^{\alpha} - k'\right) + \beta\left[C\ln(A) + D\ln(k') + E\right]
$$

First Order Condition

$$
\begin{aligned}
&-\frac{1}{(Ak^{\alpha} - k')} + \frac{\beta D}{k'} = 0 \\
&\Rightarrow k' = \beta D A k^{\alpha} - \beta D k' \\
&\Rightarrow k'(1 + \beta D) = \beta D A k^{\alpha} \\
&\Rightarrow \boldsymbol{k' = \frac{\beta D A}{1 + \beta D} k^{\alpha}}
\end{aligned}
$$

With our constraint $k' = Ak^{\alpha} - c$

$$
\begin{aligned}
&\Rightarrow c = Ak^{\alpha} - k' \\
&\Rightarrow c = Ak^{\alpha} - \frac{\beta D A}{1 + \beta D} k^{\alpha} \\
&\Rightarrow \quad = \left(\frac{1 + \beta D}{1 + \beta D}\right) Ak^{\alpha} - \frac{\beta D A}{1 + \beta D} k^{\alpha} \\
&\Rightarrow \quad = \boldsymbol{\frac{Ak^{\alpha}}{1 + \beta D}}
\end{aligned}
$$

We know that,

$$
C\ln(A) + D\ln(k) + E = \max_{k'} \ln\left(Ak^{\alpha} - k'\right) + \beta\left[C\ln(A) + D\ln(k') + E\right]
$$

So let's plus in $c = \frac{Ak^{\alpha}}{1+\beta D}$ & $k' = \frac{\beta D A}{1+\beta D} k^{\alpha}$

$$
C\ln(A) + D\ln(k) + E = \ln\left(\frac{Ak^{\alpha}}{1 + \beta D}\right) + \beta\left[C\ln(A) + D\ln\left(\frac{\beta D A}{1 + \beta D} k^{\alpha}\right) + E\right]
$$

$$
\begin{aligned}
\Rightarrow C\ln(A) + D\ln(k) + E & \\
&= \alpha\ln k + \ln A - \ln(1 + \beta D) + \beta C ln\, A + \alpha\beta D \ln k + \beta D ln\,(\beta D A) - \beta D ln\,(1 + \beta D)
\end{aligned}
$$

And I think that works out to the following:

$$\Rightarrow C = \frac{1}{(1-\beta)(1-\alpha\beta)} \quad , \quad D = \frac{\alpha}{1-\alpha\beta} \quad , \quad E = \mathcal{LOL}$$