Problem Set 3 - Sample Answers

1. a) To write this as a dynamic programming problem, start by expressing the objective function relative to autarky as

\[ U(v_0) = \max \left\{ u(c_0) - u(y_0) + E_0 \sum_{t=1}^{\infty} \beta^t (u(c_t) - u(y_t)) \right\} \]

subject to

\[ y_0 - c_0 + E_0 \sum_{t=1}^{\infty} \beta^t (y_t - c_t) \geq v_0, \]

and

\[ u(c_t) - u(y_t) + E_t \sum_{j=1}^{\infty} \beta^j (u(c_{t+j}) - u(y_{t+j})) \geq 0 \]

for each state.

Now, express the objective function as

\[ u(c_t) - u(y_t) + \sum_{s=1}^{S} \pi_s \beta^s U^s (v_{t+1}^s). \]

The dynamic programming problem is

\[ U(v_t) = \max_{c_t, \{v_{t+1}^s\}} \left\{ u(c_t) - u(y_t) + \sum_{s=1}^{S} \pi_s \beta^s U^s (v_{t+1}^s) \right\} \]

subject to

\[ y_t - c_t + \sum_{s=1}^{S} \pi_s \beta^s v_{t+1}^s \geq v_t, \]

and

\[ U^s (v_{t+1}^s) \geq 0 \]

for each \( s \).

b) Use the Lagrangian,

\[ L = u(c_t) - u(y_t) + \sum_{s=1}^{S} \pi_s \beta^s U^s (v_{t+1}^s) + \lambda_t \left( y_t - c_t + \sum_{s=1}^{S} \pi_s \beta^s v_{t+1}^s - v_t \right) + \sum_{s=1}^{S} \varphi_{t+1}^s \pi_s \beta^s U^s (v_{t+1}^s), \]

to get the first-order conditions

\[ u'(c_t) - \lambda_t = 0 \]

\[ (1 + \varphi_{t+1}^s) U'_t (v_{t+1}^s) + \lambda_t = 0 \]

\( \varphi_{t+1}^s \geq 0 \) and \( \varphi_{t+1}^s U^s (v_{t+1}^s) = 0 \)

for each \( s \),

and the envelope condition,

\[ U'(v_t) = -\lambda_t. \]

Combining, you get the Euler condition,

\[ u'(c_t) = (1 + \varphi_{t+1}^s) u'(c_{t+1}^s). \]

c) The Euler condition tells us that \( u'(c_t) \geq u'(c_{t+1}^s) \) for each state \( y_{t+1} \in \{y^1, ..., y^S\} \). This holds with equality if \( U^s (v_{t+1}^s) > 0 \). Thus, \( c_t \geq c_{t+1} \) so that consumption is monotonically increasing. Whenever,
$c_{t+1} > c_t$, $U^s (v^s_{t+1}) = 0$ so that $u(c^S_{t+1}) - u(y^s) + \sum_{s=1}^{S} \pi_s \beta U^s (v^s_{t+1}) = 0.$

To characterize the equilibrium further, start in $y^S$. In this state, define $c^S$ by

$$u(c^S) - u(y^S) + \sum_{t=1}^{\infty} \beta^t \sum_{s=1}^{S} \pi_s (u(c^S) - u(y^s)) = u(c^S) - u(y^S) + \frac{\beta}{1-\beta} (u(c^S) - Eu(y^s)) = 0$$

where this calculation uses the Euler condition’s implication that $c^S_{t+1} = c^S$ for all $s$ if $c_t = c^S$. We can next solve for $c^{S-1}$ by observing that

$$u(c^{S-1}) - u(y^{S-1}) + \beta \sum_{s=1}^{S} \pi_s U^s (v^s_{t+1}) = 0$$

where $v^s_{t+1}$ is the optimal value of the state variable when $c_t = c^{S-1}$. Use the notation, $U^S_{t+1}$, to denote the solution when $c_t = c^{S-1}$. The Euler condition implies that $U^S_{t+1} = U^{S-1}_{t+1} = 0$. This implies that

$$U^S_{t+1} = U^{S-1}_{t+1} + u(y^{S-1}) - u(y^s) = u(y^{S-1}) - u(y^s).$$

This allows us to calculate $c^{S-1}$ from

$$u(c^{S-1}) - u(y^{S-1}) + \sum_{s=1}^{S} \pi_s U^s (v^s_{t+1}) = u(c^{S-1}) - u(y^{S-1}) + \beta \sum_{s=1}^{S-1} \pi_s (u(y^{S-1}) - u(y^s)) = 0.$$ 

The $c^s$ are monotonically increasing in $s$. After these lower bounds on consumption for each state are found, the participation constraint for the risk-neutral agent is applied to determine the initial consumption, $c_0$.

2. a) The lower bounds for $w$ for the two-state solution are given by

$$u(w^2) = (1-\beta) u(y^2) + \beta Eu(y^s)$$

and

$$w^1 = y^1.$$ 

The initial consumption satisfies

$$y_0 - w_0 + E \sum_{t=1}^{\infty} \beta^t (y_t - w_t) = 0.$$ 

Let’s assume $v_0 = 0$ for simplicity. If $y_0 = y^2$, then

$$y^2 - w_0 + E \sum_{t=1}^{\infty} \beta^t (y_t - w_0) = 0 \Rightarrow w_0 = (1-\beta) y^2 + \beta E y_t$$

and if $y_0 = y^1$, then

$$y^1 - w_0 + \frac{\beta}{1-\beta} E y_t - \sum_{t=1}^{\infty} \beta^t \pi^t w_0 - (1-\pi) \sum_{t=1}^{\infty} \beta^t \pi^{t-1} \frac{1}{1-\beta} w^2 = 0$$

$$\Rightarrow w_0 = (1-\beta) y^1 + \beta E y_t$$

if $w_0 \leq w^2$, or

$$w_0 = (1-\beta) y^1 + \beta E y_t$$

if $w_0 \geq w^2$. $\pi$ is the probability state 1 occurs. Which of these two cases occurs depends on the discount factor and the distribution of $y$. 

b) The answer is just a description of the results above. If \( y_0 = y^1 \) and \( w_0 \leq w^2 \), then labor earnings rise monotonically with productivity, \( y \). Once \( y_t = y^2 \), wages are constant.

c) You should note that \( w_0 > y^1 \) and that \( w^2 < y^2 \) imply that the unconditional variance of labor earnings is less than the variance of productivity. The conditional variance of earnings on \( w_{t-1} = w^2 \) is zero.

d) If the worker can quit at time \( t \) when \( y_t = y^2 \) and work on the spot market until she chooses to take a new long-term implicit wage contract, then she will always take the spot wage when \( y_t = y^2 \). An employer will never offer a higher wage than \( y^1 \) in state 1 because he will never recover the cost of paying a wage \( w \) greater than \( y^1 \) in this state. The implicit contract fails because there is no enforcement.

e) In this case, the worker can only take a contract. A new contract offered in state 2 will pay the constant wage,

\[
\bar{w} = (1 - \beta) y^2 + \beta E_y^t
\]

which is at least as great as \( w^2 \). A new contract in state 1 will pay an initial wage

\[
w^0 = (1 - \beta_\pi) \left( y^1 + \frac{\beta}{1 - \beta} E_y^t \right) - (1 - \pi) \frac{\beta}{1 - \beta} w^2
\]

which is no greater than \( w^2 \) or

\[
\hat{w} = (1 - \beta) y^1 + \beta E_y^t
\]

if \( (1 - \beta) y^1 + \beta E_y^t > w^2 \).

First, when \( (1 - \beta) y^2 + \beta E_y^t > w^2 \), then every worker who accepted a contract first when \( y = y^2 \) will not gain (or lose) by taking a new contract in state 2. This worker would lose by quitting and taking a new contract in state 1. A worker who accepted a contract first when \( y = y^1 \) can only gain by quitting and taking a new contract in state 2. Why? Because either her wage equals \( \bar{w} \) in state 2 or equals \( (1 - \beta) y^1 + \beta E_y^t \) if \( (1 - \beta) y^1 + \beta E_y^t > w^2 \). But, \( (1 - \beta) y^1 + \beta E_y^t < (1 - \beta) y^2 + \beta E_y^t \) (by assumption, \( y^1 < y^2 \)). Thus, only if \( \bar{w} = w^2 \) does the worker fail to gain by defecting from a contract taken in state 1.

To find a solution, we need to impose a new self-enforcement constraint which is that

\[
u(c_t) + \beta E U (v_{t+1}) \geq \frac{1}{1 - \beta} u(\bar{w})
\]

for \( y_t = y^2 \). For the example, the solution for wages in period 0 in state 1 is just

\[
w^0 = (1 - \beta_\pi) \left( y^1 + \frac{\beta}{1 - \beta} E_y^t \right) - (1 - \pi) \frac{\beta}{1 - \beta} \max \{ \bar{w}, w^2_t \},
\]

where \( w^2_t \) is the new solution for \( w^2 \) from

\[
\frac{1}{1 - \beta} u \left( w^2_t \right) = \frac{1}{1 - \beta_\pi} u \left( w^0 \right) + (1 - \pi) \frac{\beta}{1 - \beta_\pi} u \left( w^2_t \right).
\]

We can see that \( w^0 \leq w^0 \) and . Thus, wage offers in state 1 will be lower because employers lose part of the compensating gain in state 2. Risk sharing is not eliminated but it is reduced by relaxing the cost of quits.

3. a)

\[
\max_{c_t, v_{t+1}^q} U_t (v) = u(c_t) - u(y_t) + \beta \sum_{s=1}^{S} \pi q U^q (v_{t+1}^q)
\]
subject to
\[
(y_t - c_t) + \beta \sum_{q=1}^{s} \pi_q v^q_{t+1} \geq v,
\]
\[
U^q (v^q_{t+1}) \geq 0 \quad \text{and} \quad v^q_{t+1} \geq 0 \quad \text{for each } s.
\]

For two states,
\[
\max_{c_t,v^1_{t+1},v^2_{t+1}} U_t (v) = u(c_t) - u(y_t) + \beta \left( \pi U^1 (v^1_{t+1}) + (1 - \pi) U^2 (v^2_{t+1}) \right)
\]
subject to
\[
(y_t - c_t) + \beta \left( \pi v^1_{t+1} + (1 - \pi) v^2_{t+1} \right) \geq v,
\]
\[
U^1 (v^1_{t+1}) \geq 0 \quad U^2 (v^2_{t+1}) \geq 0 \quad v^1_{t+1} \geq 0 \quad \text{and} \quad v^2_{t+1} \geq 0.
\]

b) The Lagrangian is
\[
L = u(c_t) - u(y_t) + \beta \left( \pi U^1 (v^1_{t+1}) + (1 - \pi) U^2 (v^2_{t+1}) \right)
\]
\[
+ \lambda_t \left( (y_t - c_t) + \beta \left( \pi v^1_{t+1} + (1 - \pi) v^2_{t+1} \right) - v \right)
\]
\[
+ \beta \pi \varphi^1_{t+1} U^1 (v^1_{t+1}) + \beta (1 - \pi) \varphi^2_{t+1} U^2 (v^2_{t+1}) + \beta \pi \psi^1_{t+1} v^1_{t+1} + \beta (1 - \pi) \psi^2_{t+1} v^2_{t+1}.
\]
The first-order conditions,
\[
u' (c_t) - \lambda_t = 0
\]
\[
U^1i (v^1_{t+1}) (1 + \varphi^1_{t+1}) + \lambda_t + \psi^1_{t+1} = 0
\]
\[
U^2i (v^2_{t+1}) (1 + \varphi^2_{t+1}) + \lambda_t + \psi^2_{t+1} = 0
\]
and the envelope condition,
\[
U^i_t (v) = -\lambda_t
\]
lead to the Euler conditions,
\[-\lambda^1_{t+1} (1 + \varphi^1_{t+1}) + \lambda_t + \psi^1_{t+1} = 0 \quad \Rightarrow \quad u' (c_t) = u' (c^1_{t+1}) (1 + \varphi^1_{t+1}) - \psi^1_{t+1}
\]
and
\[-\lambda^2_{t+1} (1 + \varphi^2_{t+1}) + \lambda_t + \psi^2_{t+1} = 0 \quad \Rightarrow \quad u' (c_t) = u' (c^2_{t+1}) (1 + \varphi^2_{t+1}) - \psi^2_{t+1}.
\]
The complementary slackness conditions,
\[
\varphi^1_{t+1} U^1 (v^1_{t+1}) = 0 \quad \varphi^2_{t+1} U^2 (v^2_{t+1}) = 0 \quad \psi^1_{t+1} v^1_{t+1} = 0 \quad \text{and} \quad \psi^2_{t+1} v^2_{t+1} = 0
\]
allow you to observe that
\[
u' (c_t) = u' (c^1_{t+1}) \quad \text{if} \quad U^1 (v^1_{t+1}) > 0 \quad \text{and} \quad v^1_{t+1} > 0
\]
\[
u' (c_t) = u' (c^1_{t+1}) (1 + \varphi^1_{t+1}) \quad \text{if} \quad U^1 (v^1_{t+1}) = 0
\]
\[
u' (c_t) = u' (c^1_{t+1}) - \psi^1_{t+1} \quad \text{if} \quad v^1_{t+1} = 0
\]
and similarly for state 2. Thus, consumption is smoothed between \( t \) and \( t + 1 \) if the self-enforcement constraints do not bind in \( t + 1 \) and it is smoothed as much as possible if the constraints do bind. This statement holds separately for states 1 and 2.
c) We have that

\[ u' (c_t) = u' (c_{t+1}^1) \text{ if } U^1 (v_{t+1}^1) > 0 \quad \text{and} \quad v_{t+1}^1 > 0 \]
\[ u' (c_t) \geq u' (c_{t+1}^1) \text{ if } U^1 (v_{t+1}^1) = 0 \]
\[ u' (c_t) \leq u' (c_{t+1}^1) \text{ if } v_{t+1}^1 = 0. \]

In the steady state of this model, \( y^1 < c^1 \leq c^2 < y^2 \) and \( c_t = c^1 \) or \( c_t = c^2 \) for all \( t \). Further, if \( c^1 \neq c^2 \) then either \( v_{t+1}^1 = 0 \) or \( U^2 (v_{t+1}^2) = 0 \) or both. Labor earnings are less variable in the implicit wage contract than in the spot labor market. Wages vary less than productivity smoothing worker consumption.

d) To find \( \bar{c}^1 \) and \( \bar{c}^2 \), use the self-enforcement constraints:

\[ u (\bar{c}^1) - u (y^1) + \beta (\pi U_{t+1}^1 + (1 - \pi) U_{t+1}^2) = 0 \]
\[ (y^1 - \bar{c}^1) + \beta (\pi v_{t+1}^1 + (1 - \pi) v_{t+1}^2) = 0 \]

The Euler condition tells us that \( U_{t+1}^1 = 0 \) if \( c_t = \bar{c}^1 \) then \( U_{t+1}^1 = 0 \),

\[ u (\bar{c}^1) - u (y^1) + \beta (1 - \pi) U_{t+1}^2 = 0 \]

and \( U_{t+1}^2 = 0 \) by contradiction - if \( U_{t+1}^2 > 0 \), then \( u' (c_t) \leq u' (c_{t+1}^2) \) which implies \( \bar{c}^1 \geq c_{t+1}^2 \) and \( \bar{c}^1 < y^1 \).

Thus, \( \bar{c}^1 = y^1 \).

For \( c_t = \bar{c}^1 \), the Euler condition implies that \( v_{t+1}^1 = 0 \),

\[ (y^1 - \bar{c}^1) + \beta (1 - \pi) v_{t+1}^2 = 0 \]

and for \( c_t = \bar{c}^2 \),

\[ u (\bar{c}^2) - u (y^2) + \beta \pi U_{t+1}^2 = 0. \]

Observe that if any smoothing is possible, then \( v_{t+1}^2 > 0 \) and \( U_{t+1}^1 > 0 \) is these last two expressions. Otherwise, \( U \) and \( v \) would be zero in both states in all equilibria (autarky is the only equilibrium). That means that \( \bar{c}^1 > y^1 \) and \( \bar{c}^2 < y^2 \).

4. a) Use the last two equations from problem 3. Observe that if \( \bar{c}^1 = \bar{c}^2 = c \) then

\[ v_{t+1}^2 = y^2 - \bar{c}^2 + \frac{\beta}{1 - \beta} \left( \pi (y^1 - \bar{c}^1) + (1 - \pi) (y^2 - \bar{c}^2) \right) = \frac{\beta \pi}{1 - \beta} (y^1 - c) + \frac{1 - \beta \pi}{1 - \beta} (y^2 - c) \]

and

\[ U_{t+1}^1 = u (\bar{c}^1) - u (y^1) + \frac{\beta}{1 - \beta} \left( \pi (u(c) - u(y^1)) + (1 - \pi) (u(\bar{c}^2) - u(y^2)) \right) \]
\[ = \frac{1 - \beta (1 - \pi)}{1 - \beta} (u(c) - u(y^1)) + \frac{\beta (1 - \pi)}{1 - \beta} (u(c) - u(y^2)). \]

Now, just substitute into those last two conditions:

\[ \left( 1 + \beta (1 - \pi) \frac{\beta \pi}{1 - \beta} \right) (y^1 - c) + \beta (1 - \pi) \frac{1 - \beta \pi}{1 - \beta} (y^2 - c) = 0 \]

and

\[ \beta \pi \frac{1 - \beta (1 - \pi)}{1 - \beta} (u(c) - u(y^1)) + \left( 1 + \beta \pi \frac{\beta (1 - \pi)}{1 - \beta} \right) (u(c) - u(y^2)) = 0. \]

These two equations solve for \( c \) and \( \beta \) simultaneously. The solution is \( \beta \).
b) If \( \beta < \overline{\beta} \), these conditions cannot both be satisfied and one of the left-hand side expressions will be negative, violating a self-enforcement constraint. For example, use the conditions 

\[
(y^1 - \overline{\pi}^1) + \beta (1 - \pi) \nu_{t+1}^2 = 0
\]

and 

\[
u_{t+1}^2 - u(y^2) + \beta \pi U_t^1 = 0, \]

again. Suppose, \( \beta = \overline{\beta} \) and then \( \beta \) is decreased. Holding the continuation values constant, \( y^1 - \overline{\pi}^1 \) and \( \nu_{t+1}^2 - u(y^2) \) must each increase to keep the self-enforcement constraints satisfied. This can only happen if \( \overline{\pi}^1 < \overline{\nu}^2 \).

c) The point is that if \( \beta > \overline{\beta} \), you have complete smoothing in the steady state. The steady-state self-enforcement constraints may not bind in either state because \( \overline{\pi}^1 > \overline{\nu}^2 \): 

\[
\left(1 + \beta (1 - \pi) \frac{\beta \pi}{1 - \beta} \right) (y^1 - c) + \beta (1 - \pi) \frac{1 - \beta \pi}{1 - \beta} (y^2 - c) \geq 0
\]

and 

\[
\beta \pi \frac{1 - \beta (1 - \pi)}{1 - \beta} (u(c) - u(y^1)) + \left(1 + \beta \pi \frac{\beta (1 - \pi)}{1 - \beta} \right) (u(c) - u(y^2)) \geq 0.
\]

There is a range of solutions for the steady-state consumption given by the interval \([\overline{\nu}^1, \overline{\nu}^2] \).

d) Where consumption ends up in the interval \([\overline{\nu}^1, \overline{\nu}^2]\) depends on the initial division of surplus. For example, if the worker gets all the surplus (employers are competitive), then if the initial spot wage \( y_0 \) equals \( y^1 \) then \( c_0 = \overline{\nu}^1 \) and this is the constant wage forever. If \( y_0 \) equals \( y^2 \), then \( c_0 = \overline{\nu}^2 = y^2 \). Although the contract does not start in the steady state, the eventual steady-state consumption is \( \overline{\nu}^1 \). The steady state is attained the first time \( y \) equals \( y^1 \).

If the employer is a monopsonist, the steady-state wage is \( \overline{\nu}^2 \) and the steady state is attained the first time \( y \) equals \( y^2 \).