Problem Set 2 - Sample Answers

1. a) Ans. The government seeks to maximize

\[ E_0 \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t Y_t f(\tau_t) \]

with respect to \( \tau_t \) for all \( t \geq 0 \), subject to the budget identity,

\[ B_{t+1} = (1 + r) B_t + G_t - \tau_t Y_t \quad (1) \]

and the solvency constraint,

\[ \lim_{T \to \infty} E_t \left( \frac{1}{1+r} \right)^{t+T} B_{t+T+1} \leq 0 \]

given initial debt, \( B_0 \).

The first-order condition can be found by writing out the Lagrangian for this problem where \( \lambda_t \) is the multiplier for the budget identity (1). The conditions are

\[ \left( \frac{1}{1+r} \right)^t f'(\tau_t) = \lambda_t \quad \text{and} \quad \lambda_t = (1 + r) E_t \lambda_{t+1} \]

which combine to

\[ f'(\tau_t) = E_t f'(\tau_{t+1}) \]

by eliminating \( \lambda_t \) and \( \lambda_{t+1} \). Another necessary condition is the transversality condition,

\[ \lim_{T \to \infty} E_t \left( \frac{1}{1+r} \right)^{t+T} f'(\tau_{t+T}) B_{t+T+1} = 0 \]

which implies the solvency constraint holds with equality.

b) Ans. Start by integrating the budget identity and using the solvency condition to get the government budget constraint,

\[ (1 + r) B_0 + E_0 \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t (G_t - \tau_t Y_t) = 0. \]

Next, solve for \( Y_t \) as

\[ Y_t = (1 - \rho) \sum_{i=0}^{t-1} \rho^i \bar{Y} + \rho^t Y_0 + \sum_{i=1}^{t} \rho^{t-i} v_i \quad \text{for} \ t > 0, \]

which becomes

\[ Y_t = \bar{Y} + \rho^t (Y_0 - \bar{Y}) + \sum_{i=1}^{t} \rho^{t-i} v_i. \]

Substitute this and the dynamic equation \( G_t \) into the budget constraint,

\[ (1 + r) B_0 + G_0 - \tau_0 Y_0 + E_0 \sum_{t=1}^{\infty} \left( \frac{1}{1+r} \right)^t \left( G_t + u_t + (g - \tau_t) \left( \bar{Y} + \rho^t (Y_0 - \bar{Y}) + \sum_{i=1}^{t} \rho^{t-i} v_i \right) \right) = 0 \]

and use the law of iterated expectations, \( E_{t-1} u_t = 0 \) to simplify to
(1 + r) B_0 + G_0 - \tau_0 Y_0 = \frac{1}{1 + r - \rho} g (Y_0 - \bar{Y}) - E_0 \sum_{i=1}^{\infty} \left( \frac{1}{1 + r} \right)^t \tau_t \left( \bar{Y} + \rho^t (Y_0 - \bar{Y}) + \sum_{i=1}^{t} \rho^t Y_{i} \right) = 0, \\
(2)

where you could write the primary deficit at t = 0 as G_0 - \tau_0 Y_0 = \bar{G} + (g - \tau_0) Y_0 + u_0.

c) Ans. Substituting for f' (\tau_t) in the first-order condition gives \tau_t = E_t \tau_{t+1}. Substituting into the budget constraint (2) and using the law of iterated expectations again we have

(1 + r) B_0 + G_0 - \tau_0 Y_0 + \frac{1}{1 + r - \rho} g (Y_0 - \bar{Y}) + \rho \tau_0 (Y_0 - \bar{Y}) + E_0 \sum_{i=1}^{\infty} \left( \frac{1}{1 + r} \right)^t \tau_t \sum_{i=1}^{t} \rho^t Y_{i} = 0.

The last term contains the covariances. Using the law of iterated expectations once more, you can see that

$$E_0 \sum_{i=1}^{\infty} \left( \frac{1}{1 + r} \right)^t \tau_t \sum_{i=1}^{t} \rho^t Y_{i} = \frac{1}{1 - \rho} E_0 \sum_{i=1}^{\infty} \left( \frac{1}{1 + r} \right)^t \tau_t \sum_{i=1}^{t} \rho^t Y_{i} = 0$$

because \tau_t = E_t \tau_{t+1} and E_t v_{t+1} = 0. You need to ignore the cov \{v_t, \tau_t\} - it does not drop out.

d) Ans. Dropping the covariance term and rearranging you get an expression for tax revenue at date 0 given by

$$\tau_0 \left( \frac{1}{1 + r - \rho} Y + \frac{r}{1 + r - \rho} Y_0 \right) = (1 + r) B_0 + \frac{1 + r}{1 + r - \rho} g (Y_0 - \bar{Y}) + u_0$$

which rearranges to

$$\tau_0 \left( \frac{1 - \rho}{1 + r - \rho} Y + \frac{r}{1 + r - \rho} Y_0 \right) = (r B_0 + \bar{G}) + \frac{r}{1 + r - \rho} g (Y_0 - \bar{Y}) + \frac{r}{1 + r - \rho} u_0$$

At time t, this is the same:

$$\tau_t \left( \frac{1 - \rho}{1 + r - \rho} Y + \frac{r}{1 + r - \rho} Y_t \right) = (r B_t + \bar{G}) + \frac{r}{1 + r - \rho} g (Y_t - \bar{Y}) + \frac{r}{1 + r - \rho} u_t. \quad (3)$$

The left-hand side of (3) is the tax rate times permanent income at date t. The right-hand side is permanent government expenditure at time t. A positive realization of u_t is a transitory expenditure increase. This leads to a rise in tax rate by the annuity value, \frac{r}{1 + r} u_t, of this increase divided by permanent income (the permanent value of the tax base). A positive realization of v_t raises permanent income by

$$\frac{r}{1 + r - \rho} v_t$$

and expenditures by

$$\frac{r}{1 + r - \rho} v_t.$$

2. a) Ans. The post-tax rental rate of return always equals \rho^* and households receive a gross return on financial assets equal to \rho^* = 1 + r^* - \delta. The marginal productivity of capital \rho_{k_t} (k_t, n_t) = r_t and \rho_{k_t} = r^* + \delta. The budget identity for the government is

$$b_t + g_t = \tau_t k_t \rho_{k_t} (k_t, n_t) + \tau_t w_t n_t + \frac{b_{t+1}}{\rho_t}$$

$$= \rho_{k_t} (k_t, n_t) - (\rho^* k_t - \bar{w} t n_t) + \frac{b_{t+1}}{\rho_t}.$$
for \( \bar{w}_t = (1 - \tau^*_t) w_t \). The household budget identity is
\[
c_t + \frac{b^d_{t+1}}{R^*_t} + k^d_{t+1} = \bar{w}_t n_t + b^d_t + (1 + r^*_t - \delta) k^d_t.
\]

b) Ans. Start with the household’s problem to derive the first-order conditions. The household maximizes
\[
\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t) \text{ with respect to } \{c_t, n_t, k_{t+1}, k^d_t, b^d_t \}_{t=0}^{\infty},
\]
subject to the household budget identity and the solvency condition for the household. The necessary conditions are
\[
\bar{w}_t u_c(c_t, 1 - n_t) = u_t(c_t, 1 - n_t),
\]
and the transversality condition. The arbitrage condition is \( R^*_t = 1 + r^*_t - \delta \).

The budget identity for the economy is
\[
c_t + k^d_{t+1} + g_t = w_t n_t + k^d_t F_k(k_t, n_t) + (1 - \delta) k^d_t + \tau^*_t F_k(k_t, n_t) \left( k_t - k^d_t \right)
\]
(note, there was a typo in the problem set - it contained the term \( r^*_t k^d_t \) instead of \( r^*_t k^d_t \)) which can be rewritten
\[
c_t + k^d_{t+1} + g_t = F(k_t, n_t) + (1 - \delta) k_t - (1 + r^*_t - \delta) \left( k_t - k^d_t \right) \tag{4}
\]
This has the interpretation that gross output, \( F(k_t, n_t) + (1 - \delta) k_t \), minus the share paid to foreign owners of capital, \( (1 + r^*_t - \delta) \left( k_t - k^d_t \right) \), equals the sum of domestic consumption, investment and government expenditure. My version uses an assumption made in the problem set that all government debt is held by domestic residents - this does not matter because bonds and capital are perfect substitutes.

The optimization problem leads to the Lagrangian,
\[
L = \sum_{t=0}^{\infty} \beta^t \left[ u(c_t, 1 - n_t) + \psi_t \left( F(k_t, n_t) - r^*_t k_t - w_t n_t + \frac{b^d_{t+1}}{R^*_t} - b_t - g_t \right) 
+ \theta_t \left( F(k_t, n_t) + (1 - \delta) k_t - (1 + r^*_t - \delta) \left( k_t - k^d_t \right) - c_t - k^d_{t+1} - g_t \right) 
+ \mu_{1t} (u_t(c_t, 1 - n_t) - \bar{w}_t u_c(c_t, 1 - n_t)) 
+ \mu_{2t} (u_c(c_t, 1 - n_t) - R^*_t \beta u_c(c_{t+1}, 1 - n_{t+1})) \right].
\]

c) Ans. The first-order condition with respect to \( k_{t+1} \),
\[
\psi_{t+1} \left( F_k(k_{t+1}, n_{t+1}) - r^*_k \right) + \theta_{t+1} \left( F_k(k_{t+1}, n_{t+1}) - r^*_k \right) = 0,
\]
implies that \( F_k(k_{t+1}, n_{t+1}) = r^*_k \). Since the first-order condition for profit maximization is
\[
\left( 1 - \tau^*_k \right) F_k(k_{t+1}, n_{t+1}) = r^*_k,
\]
the optimal source-based capital income tax rate for the small country is zero. You may also notice that the first-order condition with respect to \( k^d_{t+1} \) is
\[
\theta_{t+1} \left( 1 + r^*_t - \delta \right) = \theta_t.
\]

d) Ans. The point here is recognize that in a closed economy, the current capital stock cannot be adjusted immediately. The effects of a capital income tax on the stock of capital occur in anticipation of the future
taxation of capital. The effect on investment depends on saving behavior. In the small open economy, capital is perfectly mobile. Hence, it is perfectly elastic in supply. For the model used, capital $k_t$ can move at time $t$. Even if we restrict mobility such that capital can only move abroad next period perfectly elastically, the optimal capital income tax is still zero except, if allowed, in period 0.

See the end of the page for the primal approach to this problem!

3. a) Ans. The household maximizes $\sum_{t=0}^{\infty} \beta^t u (c_t, 1 - n_t)$ with respect to $\{c_t, n_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}$ subject to the household budget identity,

$$(1 + \tau_t^c) c_t + \frac{b_{t+1}}{R_t} + k_{t+1} = (1 - \tau_t^n) w_t n_t + b_t + (1 + \bar{r}_t - \delta) k_t,$$

and the solvency condition for the household. The first-order conditions are

$$\beta^t u_c (t) = \lambda_t (1 + \tau_t^c)$$

$$\beta^t u_t (t) = \lambda_t (1 - \tau_t^n) w_t$$

$$\lambda_t = \beta (1 + \bar{r}_{t+1} - \delta) \lambda_{t+1}$$

and

$$\lambda_t = \beta R_t \lambda_{t+1}.$$ 

The last two conditions imply $1 + \bar{r}_{t+1} - \delta = R_t$. Defining shadow prices, we have $\lambda_t = \lambda_0 q_t^0$ which we use to eliminate the multiplier from the first two conditions:

$$\frac{\beta^t u_c (t)}{1 + \tau_t^c} = \frac{q_t^0 u_c (0)}{1 + \tau_0^c} \Rightarrow q_t^0 = \frac{\beta^t u_c (t) 1 + \tau_0^c}{u_c (0) 1 + \tau_0^c}$$

and

$$\frac{u_t (t)}{u_c (t)} (1 + \tau_t^c) = (1 - \tau_t^n) w_t \Rightarrow q_t^0 (1 - \tau_0^n) w_t = \beta^t (1 + \tau_0^c) \frac{u_t (t)}{u_c (0)}.$$ 

The budget constraint for the household is

$$\sum_{t=0}^{\infty} q_t^0 [(1 + \tau_t^c) c_t - (1 - \tau_t^n) w_t n_t] = (\bar{r}_0 + 1 - \delta) k_0 + b_0.$$ 

Now, just substitute the first-order conditions into this to get an implementability constraint:

$$\sum_{t=0}^{\infty} \beta^t (1 + \tau_0^c) [u_c (t) c_t - u_t (t) n_t] = u_c (0) [(\bar{r}_0 + 1 - \delta) k_0 + b_0].$$ 

Next, define

$$V (c_t, n_t, \psi) = u (c_t, 1 - n_t) + \psi [(1 + \tau_0^c) u_c (t) c_t - (1 + \tau_0^c) u_t (t) n_t].$$ 

The Lagrangian for the optimal tax problem is

$$L = \sum_{t=0}^{\infty} \beta^t V (c_t, n_t, \psi) + \sum_{t=0}^{\infty} \beta^t \theta_t [F (k_t, n_t) + (1 - \delta) k_t - c_t - q_t - k_{t+1}] - \psi A$$

for $A \equiv u_c (0) [(\bar{r}_0 + 1 - \delta) k_0^0 + b_0].$

The first-order conditions for an optimum are

$$V_c (t) = \theta_t, \quad t > 0.$$
\[ V_n(t) = -\theta t F_n(t), \quad t > 0, \]
\[ \theta t = \beta \theta t+1 (F_k(t + 1) + (1 - \delta)), \quad t > 0, \]
\[ V_c(0) = \theta_0 + \psi A_c \]
\[ V_n(0) = -\theta_0 F_n(0) + \psi A_n. \]

The Ramsey tax plan is the solution to the following equations:

\[ V_c(t) = \beta V_c(t + 1) (F_k(t + 1) + (1 - \delta)), \quad (5) \]
\[ V_n(t) = -V_c(t) F_n(t), \quad (6) \]
\[ V_n(0) = -(V_c(0) - \psi A_c) F_n(0) + \psi A_n, \]

the resource identity,
\[ k_{t+1} + c_t + g_t = F(k_t, n_t) + (1 - \delta) k_t \]
and the implementability condition,
\[ \sum_{t=0}^{\infty} \beta^t (1 + \tau_{0_t}) [u_c(t) c_t - u_c(t) n_t] = u_c(0) [(r_0 + 1 - \delta) k_0 + b_0]. \]

b) Ans. Assume the solution converges to a steady state. In the steady state, condition (5) becomes
\[ 1 = \beta (F_k(t + 1) + (1 - \delta)) = \beta \frac{q_t^0}{q_{t+1}^0}. \]

The first-order condition for the household with respect to consumption can be rewritten
\[ \frac{q_{t+1}^0}{q_t^0} = \frac{\beta u_c(t + 1) 1 + \tau_{t+1}}{u_c(t) 1 + \tau_{t+1}}. \]

In the steady state, \( u_c(t + 1) = u_c(t) \) and \( \frac{q_{t+1}^0}{q_t^0} = \beta \) so that the optimal consumption tax is constant, \( \frac{1 + \tau_{t+1}}{1 + \tau_{t+1}} = 1 \). The optimal consumption tax converges to a value \( \tau_c \), as does the optimal labor income tax rate, \( \tau_n \).

c) Ans. Setting the labor income tax rate as suggested, \( \tau_n = -\tau_c \), rewrite the household maximization problem and notice the budget constraint is,
\[ \sum_{t=0}^{\infty} q_t^0 [(1 + \tau_c) c_t - (1 + \tau_c) w_t n_t] = (r_0 + 1 - \delta) k_0 + b_0 \]
which is the same as
\[ \sum_{t=0}^{\infty} q_t^0 [c_t - w_t n_t] = (1 + \tau_c)^{-1} [(r_0 + 1 - \delta) k_0 + b_0]. \]

The imposition of the tax on consumption and equal rate subsidy to labor income is equivalent to a lump-sum tax on initial financial wealth. Without a binding upper limit on \( \tau_c \), the optimum is just a lump-sum tax and there is no distortion. If we want an interesting problem, we need an upper bound on \( \tau_c \). Without a binding constraint on the consumption tax rate, we observe that the government should impose such a tax system which just means an equal rate tax on all consumption - that is, both goods and leisure.
4. a) Ans. This is easiest if you use the primal approach since you have the Euler condition for the household,

\[ u_c(t) = \beta (1 + \bar{r}_{t+1} - \delta) u_c(t+1), \]

and for the Ramsey solution,

\[ V_c(t) = \beta V_c(t+1) (F_k(t+1) + (1 - \delta)) . \]

Plug in \( u_c(t) = c(t)^{-\sigma} \) and \( V_c(t) = (1 + \psi (1 - \sigma)) c(t)^{-\sigma} \). Doing so implies that

\[ \bar{r}_{t+1} = \left( 1 - \tau^k_{t+1} \right) F_k(t+1) = F_k(t+1) \]

so that the optimal tax rate, \( \tau^k_t \) is zero for \( t > 1 \).

b) Ans. I am going to drop the subscripts for time where it is clear what the arguments of a function are.

With uncertainty, the Euler condition for the Ramsey solution is

\[ V_c(s^t) = \beta \sum_{s_{t+1}} \pi(s_{t+1} | s^t) \left( V_c(s_{t+1}, s^t) \left( F_k(s_{t+1}, s^t) + (1 - \delta) \right) \right) . \]

(7)

The household Euler condition is

\[ u_c(s^t) = \beta \sum_{s_{t+1}} \pi(s_{t+1} | s^t) \left( 1 + \bar{r}(s_{t+1}, s^t) - \delta \right) u_c(s_{t+1}, s^t) \]

(8)

For the utility function, we have that

\[ \frac{V_c(s_{t+1}, s^t)}{V_c(s^t)} = \frac{u_c(s_{t+1}, s^t)}{u_c(s^t)} \]

so that we must have

\[ \sum_{s_{t+1}} \frac{\beta u_c(s_{t+1}, s^t)}{u_c(s^t)} \pi(s_{t+1} | s^t) \tau^k(s_{t+1}, s^t) F_k(s_{t+1}, s^t) = 0 \]

by combining equations (7) and (8).

Now, we use the competitive equilibrium prices,

\[ p(s_{t+1}, s^t) = \frac{q^0_{k_{t+1}}(s_{t+1}, s^t)}{q^0_k(s^t)} = \frac{\beta u_c(s_{t+1}, s^t)}{u_c(s^t)} \pi(s_{t+1} | s^t) \]

and substitute

\[ \sum_{s_{t+1}} p(s_{t+1}, s^t) \tau^k(s_{t+1}, s^t) F_k(s_{t+1}, s^t) = 0. \]

This says that the ex ante optimal capital income tax equals zero.

2.* The primal approach for Problem 2:

Integrate the budget constraint for the economy,

\[ c_t + k^d_{t+1} + g_t = w_n t + r^k_t k^d_t + (1 - \delta) k^d_t + \tau^k_t F_k(k_t, n_t) k_t \]

(where I used \( F_k(k_t, n_t) - \tau^k_t = \tau^k_t F_k(k_t, n_t) \)) using the relative prices,

\[ q^0_k = \left( R^*_{k_0} R^*_{k_1} \cdots R^*_{k_{t-1}} \right)^{-1} \quad \text{for } t > 0, \quad q^0_{k_0} = 1 \]
to get
\[ \sum_{t=0}^{\infty} q_t^0 (c_t + g_t) = \sum_{t=0}^{\infty} q_t^0 w_t n_t + (r_0^* + 1 - \delta) k_0^d + \sum_{t=0}^{\infty} q_t^0 r_t^k F_k (k_t, n_t) k_t. \]  
(9)

Then use the budget identity for the government to get
\[ \sum_{t=0}^{\infty} q_t^0 r_t^k F_k (k_t, n_t) k_t = b_0 + \sum_{t=0}^{\infty} q_t^0 g_t - \sum_{t=0}^{\infty} q_t^0 (w_t - \bar{w}_t) n_t. \]  
(10)

Now, substitute (10) into (9),
\[ \sum_{t=0}^{\infty} q_t^0 c_t + \sum_{t=0}^{\infty} q_t^0 \bar{w}_t n_t = (r_0^* + 1 - \delta) k_0^d + b_0, \]
and use the substitution of goods for leisure for the household to obtain the implementability condition,
\[ \sum_{t=0}^{\infty} \beta^t u_c (c_t, 1 - n_t) c_t - u_n (c_t, 1 - n_t) n_t \left( r_0^* + 1 - \delta \right) k_0^d + b_0 = (r_0^* + 1 - \delta) k_0^d + b_0. \]  
(11)

Just in case you wonder: yes, you could derive this from the budget identity for the household.

The Lagrangian is written using equations (4) into (11) as
\[ L = \sum_{t=0}^{\infty} \beta^t \left[ u_c (c_t, 1 - n_t) + \Phi \left( u_c (c_t, 1 - n_t) c_t - u_n (c_t, 1 - n_t) n_t - \left( r_0^* + 1 - \delta \right) k_0^d + b_0 \right) u_c (c_t, 1 - n_t) \right] \]
\[ + \sum_{t=0}^{\infty} \beta^t \theta_t \left( F_k (k_t, n_t) + (1 - \delta) k_t - (1 + r_t^* - \delta) \left( k_t - k_t^d \right) - c_t - k_{t+1}^d - g_t \right). \]

Proceed from here.