1. This problem redoes the simple tax-smoothing model more formally. Suppose that the deadweight costs of taxation are measured by 
\[ Z_t = Y_t f(\tau_t), \]
where \( \tau_t = \frac{T_t}{Y_t} \) is the tax rate and \( Y_t \) is an exogenous random variable. Output net of taxes at date \( t \) is \( Y_t - T_t \). Let the discount rate for the government be the constant \( r > 0 \), the interest rate on government debt.

   a) Write out the optimization problem for the government assuming it seeks to minimize the expected present value of the deadweight costs of taxation. Allow \( Y_t \) and government expenditure, \( G_t \), to be stochastic. Show that the first-order condition for the tax rate is given by 
\[ f'(\tau_t) = E_t f'(\tau_{t+1}). \]

   b) Let \( G_t = G + gY_t + ut \) for \( 0 < g < 1 \) and \( Y_t = (1 - \rho) Y + \rho Y_{t-1} + vt \) for \( 0 < \rho < 1 \) and \( Y > 0 \) a constant. \( u_t \) and \( v_t \) are zero-mean iid random variables. Substitute these expressions into the government’s budget constraint and simplify it. Remember to leave the tax rate \( \tau_t \) as a control variable to be determined.

   c) Let \( f(\tau_t) = a\tau_t + \frac{b}{2}\tau_t^2 \) for positive parameters, \( a \) and \( b \). Revise your first-order condition. Next, use the expression you obtained from part b and the first-order condition for the tax rate to determine the optimal tax rate at an arbitrary time \( t \) given \( Y_{t-1} \). Does the covariance between \( \tau \) and \( v \) drop out? If not, you need to make a simplifying approximation and ignore it. When you do so, what is your result for \( \tau_t \)?

   d) Finish by noting how realizations of \( u_t \) and \( v_t \) affect the tax rate at time \( t \) and the similarity to the permanent income model of consumption.

2. This is problem 15.1 of L and S with some explanation. Consider a small open economy version of the optimal capital tax problem without uncertainty. Households can acquire foreign financial claims (on capital or government debt) and can sell claims to domestic capital or government bonds to foreigners. The world rental rate on capital is given and equal to \( r_t^* \). Labor, however, is immobile across countries. The government imposes a tax on labor as before and a tax on all capital used in production regardless of its ownership. The tax on capital is given by \( \tau_k \). This called a source-based capital income tax - it is imposed on all capital income earned in production in the country. (A residence-based capital income tax is a tax on the return to household savings whether from domestic or foreign sources.)

You need to know that households earn the world rental rate of return on domestic or foreign capital, so that the household realizes a net return on capital holdings equal to \( r_t^* - \delta \). This is independent of the capital stock, \( k_t \), employed domestically. Firms will receive a return equal to \( (1 - \tau_k) F_k(k_t, n_t) = r_t^* \). You also need to know how to replace the resource constraint with a budget identity for the small open economy.
Write is as the following
\[ c_t + g_t + k_{t+1}^d = w_t n_t + \tau_t^w k_t^d + (1 - \delta) k_t^d + \tau_t^k \left( k_t - k_t^d \right) F_k(k_t, n_t) \]
where \( w_t \) is the marginal productivity of labor and \( k^d \) indicates non-government asset holdings of domestic households. The last term is the government’s income from taxing foreigners.

a) Write down the budget identity for the government with the labor income tax \( \tau_t^w \) and capital income tax \( \tau_t^k \). Show that you can write this in terms of the post-tax wage rate, \( \tilde{w}_t \), and eliminate the capital income rate, \( \tau_t^k \), by making a similar substitution that done in section 15.4 of L and S. Also, write down the household budget identity.

b) Write out the problem for finding the optimal tax rates using the budget identity for the economy written by me above and the government budget identity you wrote in part a, along with the first-order conditions for the household.

c) Now, derive the optimal capital income tax rate for this economy (follow the method in section 15.4).

d) Compare the optimal source-based income tax you derived to the closed-economy solution outside the steady state. Give an intuitive economic explanation for the difference.

3. This is problem 15.2 of L and S.

a) For this part, you redo the steps in section 15.6 for the primal approach to the optimal tax problem. For example, begin with the household’s optimization problem and derive the first-order conditions. Note the arbitrage condition for the single-period returns to government bonds and capital (that is, you do not replicate equation 15.6.4). Next, substitute these to rewrite the household’s budget constraint and note the difference between the constraint for the consumption tax problem with no capital income tax and equation 15.6.5. Continue from there.

b) Answer as stated. You use the necessary conditions derived in part a.

c) Use the hint.

4. Problem 15.3 of L and S reproduces Chamley’s original model of the optimal capital income tax. Do both parts.