Present value calculations for a long-term bond

Start with a nominal bond paying a coupon of $z for every period starting next year (in year t+1) (this is a consol):

$$\$V_t = \frac{\$z}{1+i_t} + \frac{\$z}{(1+i_t)(1+i^e_{i_{t+1}})} + \frac{\$z}{(1+i_t)(1+i^e_{i_{t+1}})(1+i^e_{i_{t+2}})} + \ldots$$

Now, let the interest rate be constant:

$$\$V_t = \frac{\$z}{1+i} + \frac{\$z}{(1+i)(1+i)} + \frac{\$z}{(1+i)(1+i)(1+i)} + \ldots$$

$$\$V_t = \frac{\$z}{1+i} + \frac{\$z}{(1+i)^2} + \frac{\$z}{(1+i)^3} + \ldots$$

A really useful way to look at the price is to notice that if you hold the bond for one period, receive the coupon $z and then sell it, the price at time t+1 equals

$$\$V_{t+1} = \frac{\$z}{1+i} + \frac{\$z}{(1+i)^2} + \frac{\$z}{(1+i)^3} + \ldots$$

which is exactly the same as $\$V_t$. This means you can write the value of the bond at time t as $\$V$

$$\$V = \frac{\$z + \$V}{1+i}$$

(The economics helps you understand this equation – you hold the bond for one period and then sell it.) A little algebra,

$$\$V - \frac{1}{1+i}\$V = \$V\left(1 - \frac{1}{1+i}\right) = \$V\left(\frac{i}{1+i}\right) = \frac{\$z}{1+i},$$

leads to

$$\$V = \frac{\$z}{i}.$$

To price a bond that pays $z for ten periods, we can subtract all the payments made after the first ten from the present value calculation. Start with the price of the ten-year bond:

$$\$V_t^{10} = \frac{\$z}{1+i} + \frac{\$z}{(1+i)^2} + \frac{\$z}{(1+i)^3} + \ldots + \frac{\$z}{(1+i)^{10}},$$

I added the superscript so you will not confuse this with the price of the consol, $V. Now, this is

$$\$V_t^{10} = \frac{\$z}{1+i} + \frac{\$z}{(1+i)^2} + \frac{\$z}{(1+i)^3} + \ldots + \frac{\$z}{(1+i)^{10}} = \$V - \left(\frac{1}{1+i}\right)^{10}\left(\frac{\$z}{1+i} + \frac{\$z}{(1+i)^2} + \frac{\$z}{(1+i)^3} + \ldots\right)$$

That is, I subtract off all the payments from year 11 on. Since

$$\$V = \left(\frac{\$z}{1+i} + \frac{\$z}{(1+i)^2} + \frac{\$z}{(1+i)^3} + \ldots\right)$$

our new equation for the price of the ten-year bond is

$$\$V_t^{10} = \frac{\$z}{1+i} + \frac{\$z}{(1+i)^2} + \frac{\$z}{(1+i)^3} + \ldots + \frac{\$z}{(1+i)^{10}} = \$V - \left(\frac{1}{1+i}\right)^{10}\$V$$

$$\$V_t^{10} = \$V\left(1 - \left(\frac{1}{1+i}\right)^{10}\right)$$

$$\$V_t^{10} = \frac{\$z}{i}\left(1 - \left(\frac{1}{1+i}\right)^{10}\right)$$