Quantum mechanics of the dynamic Kingdon trap

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Classical mechanics is not sufficient to describe ion confinement in the dynamic Kingdon trap. We compute quantum corrections to the classical pseudopotential and supplement classical stability criteria with quantum stability criteria. Considering a realistic experimental scenario we show that it is possible to operate the dynamic Kingdon trap in the quantum regime.

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I. INTRODUCTION

The Kingdon trap [1], a charged wire, is the simplest device for trapping a charged particle. Neglecting radiation damping and rest-gas collisions, particles with nonzero angular momentum and opposite charge to the one on the wire revolve around the wire like planets [2] and thus are trapped forever. The Kingdon trap is still in use today and serves, for example, as a valuable tool for lifetime measurements [3]. The classical and quantum dynamics of the Kingdon trap have been investigated in great detail [4]. The advantage of the Kingdon trap is its simplicity. Its disadvantage is the angular momentum requirement in order to achieve stable ion confinement.

A trap that does not require any angular momentum, but retains the simplicity of the original static Kingdon trap is the dynamic Kingdon trap [5–16]. It is a cylindrical capacitor with a superposition of dc and ac voltages applied between the central wire and the outer cylindrical electrode. Defining the units of length and time,

$$l_0 = \left| \frac{2Z\sigma_{\mathrm{dc}}}{\pi\epsilon_0 m\Omega^2} \right|^{1/2}, \quad t_0 = 2/\Omega, \tag{1}$$

where Z is the charge of the trapped particle, σ_{dc} is the static line charge on the central wire, ϵ_0 is the permittivity of the vacuum, m is the mass of the particle, and Ω is the operating frequency of the trap, the dimensionless equation of motion of a particle with zero angular momentum stored in the dynamic Kingdon trap is given by

$$\frac{d^2\rho}{d\tau^2} = [-1 + 2\eta\cos(2\tau)]\frac{1}{\rho}, \tag{2}$$

where $\rho = r/l_0$ is the dimensionless distance of the trapped particle from the wire and η is the dimensionless control parameter of the trap [7–9].

The dynamic Kingdon trap was first investigated theoretically and experimentally under the guidance of E. Teloy at Freiburg University in the context of two unpublished student projects [5,6]. Solving Eq. (2) numerically and following computed particle trajectories over extended periods of time, these two students showed that the theoretical concept of the trap is sound. By building an actual trap in the laboratory, they showed that the trap is experimentally feasible and capable of confining particles for a long time. In addition, using the technique of averaging [17,18], Bahr and Behre computed the pseudopotential [18]

$$U_{\mathrm{eff}}^{(\mathrm{CL})}(\rho) = \ln(\rho) + \frac{\eta^2}{4\rho^2} \tag{3}$$

of the dynamic Kingdon trap and showed that a stable potential minimum exists in the space between the wire and the outer cylinder at

$$\rho_0^{(\mathrm{CL})} = \frac{\eta}{\sqrt{2}}. \tag{4}$$

Thus they provided a physical explanation for the operating principle of the trap.

After a long hiatus in which no work was done on the dynamic Kingdon trap, the trap was independently rediscovered many years later in the context of nonlinear dynamics and chaos [7–9]. In addition to the seminal work by Bahr and Behre [5,6] it was shown that the dynamic Kingdon trap exhibits chaos already on the single-particle level [7] and that in conjunction with laser cooling [19] crystalline configurations can be obtained in the dynamic Kingdon trap [8,9]. It was also shown that three-dimensional particle confinement is achieved with the help of end caps [7], without significantly changing the equation of motion (2). The trap exhibits period-doubling bifurcations [7,9,13] and stable limit cycles [7,10]. The existence of period-doubling bifurcations has been confirmed experimentally [11]. Additionally, the dynamic Kingdon trap has already found a practical application [14].

A kicked version of the dynamic Kingdon trap was investigated in Ref. [12]. It was shown that after judiciously adjusting the parameters of the kicked trap, its classical dynamics is very close to the cw-driven dynamic Kingdon trap.

The dynamic Kingdon trap also has some surprises in store. Since the pseudopotential (3) is globally confining, one might be lead to think that the dynamic Kingdon trap confines particles for all choices of the control parameter η. This is not the case. Due to nonlinear resonances, beyond the reach of the pseudopotential approximation, control parameters exist at which the dynamic Kingdon trap is unstable [15,16].

The dynamic Kingdon trap is a close cousin of the Paul trap [20–22]. Both traps are electrodynamical in nature and their operating principle, the generation of focusing and defocusing forces due to rapidly oscillating inhomogeneous fields, is based on the Kapitza effect [17]. Despite many similarities, there are also important differences. While the single-particle dynamics of the ideal Paul trap is exactly integrable and explicitly solvable both classically [20–22] and quantum mechanically [23–25], the dynamic Kingdon trap is a chaotic system with a mixed phase space and exhibits a period-doubling scenario [7,9]. The Paul trap, too, is a chaotic system, but only if two or more particles are stored simultaneously in the trap [26,27]. Therefore the dynamic Kingdon trap is an ideal device for studying nonlinear phenomena [28] and quantum chaos [29] since it displays all of the classic chaos phenomena already on the single-particle level.

The most important difference between the Paul trap and the dynamic Kingdon trap, however, is the following. For a single stored particle the stability of the Paul trap does not depend on the initial conditions of the particle in phase space. Thus stability in the Paul trap depends only on the trap's control parameters [20–27]. In the dynamic Kingdon trap, however, trapping is achieved by placing the particle inside of a regular phase-space island, the trapping island [7–9,15,16]. Thus, in addition to the control parameter η, ion confinement in the dynamic Kingdon trap depends decisively on the initial conditions of the particle. Classically, once an ion is placed inside of the trapping island, it is confined to the island forever. Due to the finite size of Planck's constant $\hbar$, however, the quantum wave packet associated with the trapped particle has a finite width. If the quantum width of the packet is larger than the phase-space width of the trapping island, the particle's wave function "spills out" into the chaotic sea. Thus, while quantum mechanics in the single-particle Paul trap is not important for deciding whether a particle is trapped or not, it is of crucial importance in the dynamic Kingdon trap. In fact, in Sec. III, we amend the classical trapping criteria of the dynamic Kingdon trap with quantum trapping criteria.

Thus, especially at low ion temperatures, classical mechanics is not sufficient to describe the behavior of trapped particles in the dynamic Kingdon trap. Although crucial for the understanding of particle trapping in the dynamic Kingdon trap, the quantum mechanics of the dynamic Kingdon trap has never before been addressed in the literature. This paper fills the gap. It is organized in the following way. In Sec. II we derive the quantum pseudopotential and compare it to the classical pseudopotential (3). We show that the two are only approximately equal; there are important quantum corrections. In Sec. III we use the pseudopotential picture to derive our quantum trapping criteria. In Sec. IV we discuss our results. In Sec. V we summarize and conclude our paper.

II. CLASSICAL VERSUS QUANTUM PSEUDOPOTENTIAL

Trapping of a particle in the dynamic Kingdon trap can be understood with the help of the idea of a time-averaged pseudopotential [17,18]. The construction of a pseudopotential goes back to an idea of Kapitza [17]. The pseudopotential has been used extensively in the trapping literature in order to understand the trapping mechanism and to determine the characteristics of traps such as widths and depths (for a review see [22]). The classical pseudopotential is given by Eq. (3). The quantum pseudopotential is computed in the following way.

The force $\vec{F}$ acting on a particle in the dynamic Kingdon trap is derived from the potential

$$U(\vec{r},t) = \left|\frac{Z\sigma_{\mathrm{dc}}}{2\pi\epsilon_0}\right| \ln\left(\frac{r}{l_0}\right)[1 - 2\eta\cos(\Omega t)] \tag{5}$$

via $\vec{F}=-\vec{\nabla}U=-\hat{r}\,\partial U/\partial r$. We use this potential in the time-dependent Schrödinger equation for a single particle in the dynamic Kingdon trap:

$$i\hbar\,\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi(\vec{r},t) + U(\vec{r},t)\psi(\vec{r},t). \tag{6}$$

Because of the cylindrical symmetry of the trap,

$$\psi(\vec{r},t) = \Phi(r,t)e^{iM\theta}e^{iKz}, \tag{7}$$

where $M=0,\pm1,\pm2,\ldots$ is the azimuthal quantum number and $\hbar K$ is the momentum in z direction. Introducing the dimensionless variables defined in Eq. (1), the radial part of the Schrödinger equation becomes

$$i\alpha\frac{\partial\Phi(\rho,\tau)}{\partial\tau} = -\frac{\alpha^2}{2}\Delta_\rho\Phi(\rho,\tau) + [1 - 2\eta\cos(2\tau)]\ln(\rho)\Phi(\rho,\tau), \tag{8}$$

where

$$\Delta_\rho = \frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{M^2}{\rho^2} - K^2 l_0^2 \tag{9}$$

and α, the effective, dimensionless Planck constant is given by

$$\alpha = \left|\frac{\hbar\Omega\pi\epsilon_0}{Z\sigma_{\mathrm{dc}}}\right| = \frac{\hbar}{L_0}, \tag{10}$$

where

$$L_0 = \frac{1}{2}ml_0^2\Omega \tag{11}$$

is the unit of angular momentum. The second equality in Eq. (10) is particularly illuminating since it expresses the effective Planck constant α as the ratio of Planck's constant $\hbar$ and a typical system action, in this case the classical unit of angular momentum (11). Defining

$$\varphi(\rho,\tau) = \rho^{1/2}\Phi(\rho,\tau) \tag{12}$$

and

$$V(\rho) = \frac{\alpha^2}{2}\left(\frac{M^2 - 1/4}{\rho^2} + K^2 l_0^2\right) + \ln(\rho), \tag{13}$$

we obtain the one-dimensional radial Schrödinger equation

$$i\alpha\frac{\partial\varphi(\rho,\tau)}{\partial\tau} = -\frac{\alpha^2}{2}\frac{\partial^2\varphi(\rho,\tau)}{\partial\rho^2} + V(\rho)\varphi(\rho,\tau) - 2\eta\cos(2\tau)\ln(\rho)\varphi(\rho,\tau). \tag{14}$$

Define a new wave function $\phi(\rho,\tau)$ via

$$\varphi(\rho,\tau) = e^{i\eta\ln(\rho)\sin(2\tau)/\alpha}\phi(\rho,\tau). \tag{15}$$

Inserting Eq. (15) into the Schrödinger equation (14) we obtain the following Schrödinger equation for $\phi(\rho,\tau)$:

$$i\alpha\frac{\partial\phi(\rho,\tau)}{\partial\tau} = -\frac{\alpha^2}{2}\frac{\partial^2\phi(\rho,\tau)}{\partial\rho^2} + U_{\text{eff}}^{(\text{QM})}(\rho)\phi(\rho,\tau) + i\alpha\eta\sin(2\tau) \times\left[\frac{\phi(\rho,\tau)}{2\rho} - \frac{\partial\phi(\rho,\tau)}{\partial\rho}\right]\frac{1}{\rho} - \frac{\eta^2}{4\rho^2}\cos(4\tau)\phi(\rho,\tau), \tag{16}$$

where the quantum pseudopotential $U_{\text{eff}}^{(\text{QM})}$ is given by

$$U_{\text{eff}}^{(\text{QM})} = V(\rho) + \frac{\eta^2}{4\rho^2}. \tag{17}$$

In order to compare the quantum pseudopotential (17) with the classical pseudopotential (3) we set $M=0$ and $K=0$ in Eq. (13) and obtain

$$U_{\text{eff}}^{(\text{QM})} = -\frac{\alpha^2}{8\rho^2} + \ln(\rho) + \frac{\eta^2}{4\rho^2}. \tag{18}$$

Apparently $U_{\text{eff}}^{(\text{CL})}(\rho)$ and $U_{\text{eff}}^{(\text{QM})}(\rho)$ are not the same. They differ by a quantum correction $-\alpha^2/8\rho^2$. From Eq. (18) we obtain the quantum equilibrium position

$$\rho_0^{(\text{QM})} = \frac{1}{2}\sqrt{2\eta^2 - \alpha^2}. \tag{19}$$

It is different from the classical equilibrium position (4). The quantum shift in the equilibrium position is approximately given by

$$\rho_0^{(\text{QM})} - \rho_0^{(\text{CL})} \approx -\frac{\alpha^2}{4\eta\sqrt{2}}. \tag{20}$$

It may be detectable experimentally.

At $\rho_0^{(\text{QM})}$ the quantum pseudopotential can be approximated by an oscillator potential

$$U_{\text{osc}}(\rho) = U_{\text{eff}}(\rho_0^{(\text{QM})}) + \frac{1}{2}\omega_{\text{osc}}^2(\rho - \rho_0^{(\text{QM})})^2, \tag{21}$$

where

$$\omega_{\text{osc}} = \frac{2}{\sqrt{\eta^2 - \alpha^2/2}} \tag{22}$$

is the pseudo-oscillator frequency. The second-order approximation (21) to the pseudopotential is known as the pseudo-oscillator. For later considerations in connection with the experimental feasibility of operating a dynamic Kingdon trap in the quantum regime (see below) we state the pseudo-oscillator frequency (22) in SI units. It is given by

$$\omega_{\text{osc}}^{\text{SI}} = \frac{E_0\alpha\omega_{osc}}{\hbar} = \frac{\Omega}{\sqrt{\eta^2 - \alpha^2/2}}, \tag{23}$$

where

$$E_0 = \frac{1}{4}m\Omega^2 l_0^2 \tag{24}$$

is the unit of energy.

III. QUANTUM STABILITY CRITERIA

The pseudo-oscillator approximation is an excellent guide for a qualitative exploration of the quantum mechanics of the dynamic Kingdon trap. The quantum states in the pseudo-oscillator (21) are harmonic oscillator states given by

$$\varphi_n(\rho) = [2^n n!\, b\sqrt{\pi}]^{-1/2}H_n[(\rho - \rho_0^{(\text{QM})})/b] \times\exp[-(\rho - \rho_0^{(\text{QM})})^2/(2b^2)], \quad n = 0,1,2,\ldots, \tag{25}$$

where H_n are the Hermite polynomials [30] and b is the oscillator length given by

$$b = \sqrt{\frac{\alpha}{2}\sqrt{\eta^2 - \alpha^2/2}}. \tag{26}$$

The spatial and momentum widths of the wave functions (25) are given by

$$\Delta\rho_n = b\sqrt{n + 1/2}, \quad \Delta\dot{\rho}_n = \frac{\alpha}{b}\sqrt{n + 1/2}. \tag{27}$$

As a check we compute the uncertainty product of Eq. (27). It is given by $\Delta\rho_n\Delta\dot{\rho}_n = (n+1/2)\alpha \geqslant \Delta\rho_0\Delta\dot{\rho}_0 = \alpha/2$, consistent with Heisenberg's uncertainty relation.

Since the trapping island of a dynamic Kingdon trap has a finite width in ρ and $\dot{\rho}$ [7–9,15,16], not all of the wave functions (25) can possibly be good approximations of the exact quantum states, since according to Eq. (27) their widths grow with increasing n, such that for high n the corresponding wave functions will eventually spill out of the island and into the chaotic sea. This observation results in new quantum stability criteria supplementing the classical stability criterion according to which a particle is trapped as soon as its initial conditions are located inside of the trapping island.

Let us first estimate the maximum number of quantum states, N, that are supported by a trapping island of spatial width $\delta\rho$ and momentum width $\delta\dot{\rho}$. With Eq. (27) (note that the counting of n states starts at $n=0$) we obtain

$$N \approx \min\left[\left(\frac{\delta\rho}{b}\right)^2, \left(\frac{b\,\delta\dot{\rho}}{\alpha}\right)^2\right] + \frac{1}{2}. \tag{28}$$

For stable trapping at least one quantum state (the ground state φ_0) has to be supported by the trapping island. Thus we have to require

$$N \geqslant 1. \tag{29}$$

This is our first quantum stability criterion. For given α it sets limits on $\delta\rho$ and $\delta\dot{\rho}$.

For given $\delta\rho$ and $\delta\dot{\rho}$ we obtain a limit on α if we require that at least one quantum state should be trapped. For small α we approximate $b^2 \approx \alpha\eta/2$ and obtain

$$\alpha/2 < \min\left[\frac{2}{\eta}(\delta\rho)^2, \frac{\eta}{2}(\delta\dot{\rho})^2\right]. \qquad (30)$$

This is our second quantum stability criterion.

With Eq. (28) we estimate the number of quantum states confined to the trapping island at $\eta=5$ for a realistic situation. In this case the spatial width of the trapping island is $\delta\rho \approx 0.6$ and the momentum width is $\delta\dot{\rho} \approx 0.2$ [15]. Suppose we operate the dynamic Kingdon trap at $\Omega = 10^7\ \mathrm{s}^{-1}$, use $^{24}Mg^{+}$ as the trapped particle species [27], and arrange the dc and ac voltages applied to the trap such that $l_0 = 1$ mm; we find $N \approx 2 \times 10^8$. The large number of trapped quantum states indicates that under normal operating conditions classical mechanics is an excellent approximation to the trapped particle's dynamics. Moreover, under room-temperature conditions ($T = 300$ K), the thermal ratio $\hbar\omega_{\mathrm{osc}}^{\mathrm{SI}}/(kT) \approx 5 \times 10^{-8} \ll 1$, indicating that the thermal energy of the trapped particle exceeds by far the quantum energy-level spacing. However, based on current technology, it is possible to operate a dynamic Kingdon trap in the quantum regime. First we note that according to Eq. (23) the pseudo-oscillator frequency of the dynamic Kingdon trap is directly proportional to the trap frequency. This is a crucial observation in view of a possible experimental implementation of a quantum dynamic Kingdon trap. It means that the higher the trap frequency, the stiffer the pseudo-oscillator, the larger the spacing between the trap's quantum states, the better the chances for the experimental resolution of individual quantum states. For the separation of the energy levels at temperature T, for instance, we need $\hbar\omega_{\mathrm{osc}}^{\mathrm{SI}}/(kT) > 1$, which is, for instance, fulfilled with $\Omega = 10^8\ \mathrm{s}^{-1}$ and $T = 0.1$ mK. With laser cooling techniques $T = 0.1$ mK has already been reached [31,32]. All that remains to be shown is that the relatively high trap frequency ($\Omega = 10^8\ \mathrm{s}^{-1}$) does not lead to impossible conditions on trap voltages and the diameter of the wire. According to Eq. (1), $l_0 = 1$ mm, for instance, can be achieved with electric fields of about 6×10^3 V/cm, well within technical reach. Moreover, lowering the temperature by one order of magnitude allows a reduction of Ω to $10^7\ \mathrm{s}^{-1}$. Since the required electric fields are proportional to the square of the trap frequency, this lowers the required electric fields by two orders of magnitude to about 60 V/cm, well within the range of voltages used to operate ion traps [14,18,20–22,26,27,31–34]. We conclude that based on the pseudo-oscillator picture there are no physical principles that would prevent reaching the quantum regime in the dynamic Kingdon trap. We note that the quantum regime has already been reached in other types of electrodynamical traps [31–34].

IV. DISCUSSION

Of all the quantum dynamical traps the dynamic Kingdon trap is perhaps the most interesting one from a nonlinear dynamics point of view. Loaded with only a single charged particle the trap shows a mixed phase space that possesses all of the classic phase-space morphology, including regular islands and a chaotic sea. Experimentally accessible quantum chaotic systems are rare. We believe that due to its simplicity the dynamic Kingdon trap will join the small family of quantum chaos experiments. Among these the dynamic Kingdon trap is perhaps closest in spirit to the hydrogen atom in a strong microwave field [35,36]. Both systems are driven by external ac fields, and both systems show a mixed phase space.

Contrary to many other ion traps, such as the single-particle Paul trap, the single-particle dynamic Kingdon trap is not analytically solvable. The reason is the occurrence of chaos. The absence of an analytical quantum solution indicates the existence of a host of as-yet-undiscovered quantum phenomena, among which may be the existence of quantum localized states [35] or cantorus tunneling phenomena [37].

Our quantum stability criteria (29) and (30) are based on the classical intuition that once in the chaotic sea the particles are lost, because they quickly diffuse chaotically toward the trap's electrodes and discharge. However, it is well known that in many cases chaotic diffusion produces quantum localization [35]. Quantum localization is produced by a subtle, destructive quantum phase interference process which is easily destroyed by ambient noise [38]. Therefore, even if the quantum dynamic Kingdon trap should show quantum localization, quantum states trapped inside of the trapping island are expected to be much more stable over long trapping times than quantum localized states in the chaotic sea. Therefore, even in the presence of quantum localization, we expect our quantum stability criteria to retain their validity indicating the borderline between stable, island-trapped states, and fragile, quantum-localized states.

The quantum dynamic Kingdon trap provides conceptual advantages compared with many other chaotic traps. We mention the example of the two-particle Paul trap, another chaotic electrodynamical trap. To our knowledge the quantum mechanics of this trap has only been studied with the help of approximate techniques [39] or by averaging over the micromotion in the pseudopotential approximation [40]. The difficulty here stems from the fact that even if we use all the symmetries of the trap to our advantage, the phase space of the two-ion Paul trap is still irreducibly four dimensional. This may be compared with the two-dimensional phase space of the dynamic Kingdon trap (ρ and $\dot{\rho}$), a considerable simplification for analytical and experimental work.

V. SUMMARY AND CONCLUSIONS

In this paper we show that classical mechanics alone is not sufficient to understand the physics of the dynamic Kingdon trap. We show that the classical and quantum pseudopotentials differ by a small quantum correction, which may be experimentally measurable. We derive two quantum stability conditions for the dynamic Kingdon trap. For given Planck's constant the first condition, (29), sets limits on the phase-space dimensions of the trapping island. For a given trapping island the second condition, (30), sets a limit on the effective Planck constant α. Simple analytical estimates based on the pseudopotential approach show that it should be possible to operate the dynamic Kingdon trap in the quantum regime in the laboratory. Because of its simple, but nevertheless representative phase-space structure, we are convinced that the dynamic Kingdon trap has much to offer for the fields of theoretical and experimental quantum chaos research.

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