1. Average Values

We know given a set of numbers, \( \{x_1, x_2, ..., x_n\} \) how to calculate the average, or mean, it is just
\[
\frac{x_1 + x_2 + \cdots + x_n}{n}
\]
What if we have something more continuous, like trying to find the average temperature during the day, which changes continuously with respect to time, or the average velocity an object is moving. We extend this idea of an average.

**Example 1.1.** Consider the graph of a line \( y = x \), from \( a \) to \( b \), its average is about the midpoint of the line segment from \((a, a)\) to \((b, b)\), which is \((\frac{a+b}{2}, \frac{a+b}{2})\).

To extend the idea of an average value for a continuous function \( f(x) \) for \( x \in [a, b] \), we divide the interval \([a, b]\) into \( n \) equal subintervals, whose points are given by \( x_1^*, x_2^*, ..., x_n^* \). Let \( \Delta x \) be the width of these subintervals. To compute the average value of the function for these \( n \) points, it is just the usual average
\[
\frac{f(x_1^*) + f(x_2^*) + ... + f(x_n^*)}{n}
\]
This is analogous to taking sample readings. For example, if \( f \) is a function that gives us temperature, it is a periodic sampling of the temperature throughout the day. If the function is velocity, we take samplings of velocity at periodic integrals.

Since \( \Delta x = \frac{b-a}{n} \), we can rewrite this as \( n = \frac{b-a}{\Delta x} \). This gives us,
\[
\frac{f(x_1^*) + f(x_2^*) + ... + f(x_n^*)}{\frac{b-a}{\Delta x}} = \frac{1}{b-a} \left[ f(x_1^*) + f(x_2^*) + ... + f(x_n^*) \right] \Delta x
\]
\[
= \frac{1}{b-1} \sum_{i=1}^{n} f(x_i^*) \Delta x
\]
Continuing as we did to compute area between a curve and the \( x \)-axis, lets take more and more sampling points, so we take the limit as \( n \) goes
to infinity,
\[
\lim_{n \to \infty} \frac{1}{b - 1} \sum_{i=1}^{n} f(x_i^*) \Delta x = \frac{1}{b - a} \int_{a}^{b} f(x) dx
\]

**Definition 1.2.** The average value of a continuous function \( f \) is given by to infinity,
\[
\frac{1}{b - a} \int_{a}^{b} f(x) dx
\]

**Example 1.3.** (1) Show that average value of \( y = 2x \) between 0 and 4 is 4 using the midpoint and integral.
(2) Compute the average value of \( x^3 - 2 \) between -1 and 3. Answer should be \( \frac{1}{3} \).
(3) The temperature \( t \) hours after 9 am is modeled by the function
\[
T(t) = 50 + 14 \sin \left( \frac{\pi t}{12} \right)
\]
Find the average temperature during the period from 9 am to 9 p.m.

Now consider the following question. Given the average value of a continuous function \( f(x) \) on an interval, say \( f_{avg} \), can we find at which points \( x = c \) does \( f(x) = f_{avg} \)? Does such an \( c \) exist?

**Theorem 1.4.** (Mean Value Theorem for Integrals)
Suppose \( f \) is continuous on \([a,b]\), then there exists a number \( c \in [a,b] \) such that
\[
f(c) = \frac{1}{b - a} \int_{a}^{b} f(x) dx
\]
or
\[
\int_{a}^{b} f(x) dx = f(x)(b - a)
\]

**Proof.** Extra Credit (4) (10 points) \( \square \)

**Example 1.5.** If \( f \) is continuous and \( \int_{1}^{3} f(x) dx = 8 \), show that \( f \) takes on the value 4 at least once on the interval \([1,3]\).