1. Integration by Parts

Integration by parts is the analogue of the product rule from differentiation. Recall the product rule, which gives us a way of differentiating a product of two functions:

\[
\frac{d[f(x)g(x)]}{dx} = f(x) \frac{dg(x)}{dx} + \frac{df(x)}{dx} g(x) = f(x)g'(x) + f'(x)g(x).
\]

Another way of saying the same statement above is

\[
\int [f(x)g'(x) + f'(x)g(x)]dx = f(x)g(x)
\]

\[
\Leftrightarrow \int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.
\]

The last equality is the **Integration by Parts for Indefinite Integrals** equation. This can be rewritten as

\[
\int udv = uv - vdu,
\]

where \(u = f(x), \ v = g(x), \ du = f'(x)dx, \ dv = g'(x)dx,\) and using \(u\)-substitution.

**Example 1.1.** Evaluate the following integrals

1. \(\int x\cos(x)dx\)
   
   Let \(u = x\) and \(dv = \cos(x)dx\).
   
   **Answer:** \(x\sin(x) + \cos(x) + c, \) where \(c \in \mathbb{R}\).

2. \(\int \ln(x)dx\)
   
   Let \(u = \ln(x)\) and \(dv = 1dx\).
   
   **Answer:** \(x\ln(x) - x + c, \) where \(c \in \mathbb{R}\).

3. \(\int x^3e^x dx\)
   
   Let \(u = x^3\) and \(dv = e^x.\) We need to do integration by parts three times, using the analogous choices for \(u\) and \(dv\).
   
   **Answer:** \(x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c, \) where \(c \in \mathbb{R}\).

4. \(\int e^x \cos(x)dx\)
   
   Let \(u = \cos(x)\) and \(dv = e^x.\) We need to do integration by parts
twice. The second time, we make a consistent choice for \( u \) to be the trig function that shows up and \( dv = e^x \). We could have also chose \( u = e^x \) and \( dv = \cos(x) \).

**Answer:** \( e^x \cos(x) + c \), where \( c \in \mathbb{R} \).

**Integration by Parts for Definite Integrals** is analogous. Just use the Fundamental Theorem of Calculus part II,

\[
\int_a^b f(x)g'(x)\,dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x)\,dx.
\]

**Example 1.2.** (1) Evaluate the following integral.

\[
\int_0^1 \tan^{-1}(x)\,dx
\]

**Solution:** Let \( u = \tan^{-1}(x) \) and \( dv = dx \). Recall that \( \frac{du}{dx} = \frac{1}{1+x^2} \). We need to use integration by parts, and then a \( u \)-substitution to solve this integral.

**Answer:** \( \frac{\pi}{4} - \frac{1}{2} \ln|2| \).

(2) Evaluate the following:

\[
\int_4^9 \frac{\ln(y)}{\sqrt{y}}\,dy
\]

**Solution:** We first let \( u = \ln(y) \) and \( dv = y^{-1/2}dy \). **Answer:** \( 6\ln|9| - 4\ln|4| - 4 \).

2. **Partial Fraction Decomposition**

This is a technique for integrating \( \int \frac{P(x)}{Q(x)}\,dx \) where \( P(x) \) and \( Q(x) \) are both polynomials.

A partial fraction is something of the following forms:

\[
\frac{A}{ax + b} + \frac{A}{(ax + b)^2} + \frac{Ax + B}{ax^2 + bx + c} + \frac{Ax^2 + Bx + C}{(ax^2 + bx + c)^2} + \frac{Ax^2 + Bx + C}{ax^3 + bx^2 + cx + d}
\]

where \( a, b, c, d, A, B, C \in \mathbb{R} \) and \( k \in \mathbb{N} \), and the denominators cannot be factored.

If \( f(x) = \frac{P(x)}{Q(x)} \) is a rational function, and the degree of \( P(x) \) is less than the degree of \( Q(x) \), then \( f(x) \) can always be rewritten as a sum of partial fractions.

If we were able to break up \( f(x) \) as a sum of partial fractions, this makes \( \int \frac{P(x)}{Q(x)}\,dx \) easier to integrate, because partial fractions are easier to deal with.

**Example 2.1.** Find the partial fraction decomposition of the following:
(1) \( \frac{x}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} \)

(2) \( \frac{x^3 - 2x + 1}{(x^2+1)(x^2-1)} \)

(3) \( \frac{x^5 + 3x^2 + 2}{(x+2)(x-3)^2(x^2+x+1)^3} \)

**Example 2.2.** Evaluate.

\[
\int_0^1 \frac{x - 1}{x^2 + 3x + 2} \, dx
\]

**Answer:** \( \ln(27) - \ln(32) \)