1. The Definite Integral

**Theorem 1.1. (Properties of Integrals)**

1. \(\int_{a}^{b} f(x)dx = -\int_{b}^{a} f(x)dx\)
2. \(\int_{a}^{a} f(x)dx = 0\)
3. \(\int_{a}^{b} cdx = c(b - a)\) where \(c \in \mathbb{R}\)
4. \(\int_{a}^{b} [f(x) \pm g(x)]dx = \int_{a}^{b} f(x)dx \pm \int_{a}^{b} g(x)dx\)
5. \(\int_{a}^{b} cf(x)dx = c \int_{a}^{b} f(x)dx\)
6. \(\int_{a}^{b} f(x)dx + \int_{c}^{b} f(x)dx = \int_{a}^{c} f(x)dx\)
7. If \(f(x) \geq 0\) for \(a \leq x \leq b\), then \(\int_{a}^{b} f(x)dx \geq 0\)
8. If \(f(x) \geq g(x)\) for \(a \leq x \leq b\), then \(\int_{a}^{b} f(x)dx \geq \int_{a}^{b} g(x)dx\)
9. If \(m \leq f(x) \leq M\) for \(a \leq x \leq b\), then

\[m(b - a) \leq \int_{a}^{b} f(x)dx \leq M(b - a)\]

**Example 1.2.** Use the properties of integrals to evaluate

\[
\int_{0}^{1} \left(\frac{1}{2}x^2 - 5 + x\right)dx
\]

**Solution:** Recall from the first lecture that \(\int_{0}^{1} x^2 = \frac{1}{3}\) and that \(\int_{0}^{1} xdx\) is the area of a triangle of base and height both of length 1 so \(\int_{0}^{1} xdx = \frac{1}{2}\).

\[
\int_{0}^{1} \left(\frac{1}{2}x^2 - 5 + x\right)dx = \int_{0}^{1} \left(\frac{1}{2}x^2\right)dx - \int_{0}^{1} 5dx + \int_{0}^{1} xdx
\]

\[
= \frac{1}{2} \int_{0}^{1} x^2 dx - \int_{0}^{1} 5dx + \frac{1}{2}
\]

\[
= \frac{1}{2} \left(\frac{1}{3}\right) - 5 + \frac{1}{2}
\]

\[
= \frac{-13}{3}
\]
We can see why the integral gives us a negative number by looking at the area of the region that the integral represents.

Example 1.3. Use the upper/lowerbound property to estimate

\[
\int_{0}^{2} \frac{1}{1 + x^2} \, dx
\]

**Solution:** Since \( x \in [0,1] \) we have that the largest that \( x \) can be is 1 and the smallest value it can be is 0. \( \frac{1}{1+x^2} \) is also decreasing on \([0,1]\) (see the graph below).

Thus,

\[
\frac{1}{2} \leq \frac{1}{1 + x^2} \leq 1
\]

where the left inequality is obtained from \( x = 1 \) and on the right from \( x = 0 \). Using property (9) we have

\[
\frac{1}{2} \leq \int_{0}^{2} \frac{1}{1 + x^2} \, dx \leq 1
\]

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2. The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus, Part I.

**Theorem 2.1.** (Fundamental Theorem of Calculus Part I)
If \( f \) is continuous on \([a, b]\) then the function \( F \) defined by
\[
F(x) = \int_a^b f(t) \, dt \quad a \leq x \leq b
\]
is an antiderivative of \( f \), that is, \( F'(x) = f(x) \) for \( x \in (a, b) \).

**Proof.**

\[
F'(x) = \lim_{h \to \infty} \frac{F(x + h) - F(x)}{h} = \lim_{h \to \infty} \frac{\int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt}{h} = \lim_{h \to \infty} \frac{\int_x^{x+h} f(t) \, dt}{h}
\]

Recall the Extreme Value Theorem.

**Theorem 2.2.** (Extreme Value Theorem)
If \( f \) is continuous on a closed interval \([a, b]\), then \( f \) attains an absolute maximum value \( f(c) \) and an absolute minimum value \( f(d) \) at some numbers \( c \) and \( d \) in \([a, b]\). So by the EVT, there exists a \( k, l \in \mathbb{R} \) such that
\[
x \leq k, l \leq x + h
\]
where \( m = f(k) \) and \( M = f(l) \) and
\[
f(k) \leq f(t) \leq f(l)
\]
for all \( t \in [x, x + h] \). We have by property (9) that
\[
 mh \leq \int_x^{x+h} f(t) \, dt \leq Mh
\]
dividing by \( h \),
\[
m = f(k) \leq \frac{\int_x^{x+h} f(t) \, dt}{h} \leq M = f(l)
\]
Since \( h > 0 \), and \( x \leq k, l \leq x + h \), as \( h \) tends towards 0, \( k = l = x \) by the squeeze theorem. Similarly
\[
f(x) \leq \lim_{h \to \infty} \frac{\int_x^{x+h} f(t) \, dt}{h} = F'(x) \leq f(x)
\]
thus $F'(x) = f(x)$. □

Example 2.3. Find the derivative of

$$g(y) = \int_2^4 t^2 \sin t \, dt$$

Solution: $g'(y) = y^2 \sin y$.

Fundamental Theorem of Calculus Part II.

Theorem 2.4. (Fundamental Theorem of Calculus Part II)
If $f$ is continuous on $[a, b]$ then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

where $F'(x) = f(x)$ and $F(x)$ is any antiderivative of $f$.

Proof. Let $g(x) = \int_a^x f(t) \, dt$. By the FTC I, $g'(x) = f(x)$. Let $F(x)$ be any antiderivative of $f(x)$. Then $F(x) = g(x) + c$.

$$F(b) - F(a) = g(b) + c - (g(a) + c)$$

$$= g(b) - g(a)$$

$$= \int_a^b f(t) \, dt - \int_a^a f(t) \, dt$$

$$= \int_a^b f(t) \, dt.$$

Example 2.5. Evaluate $\int_1^9 \frac{x-1}{\sqrt{x}} \, dx$.

Solution:

$$\int_1^9 \frac{x-1}{\sqrt{x}} \, dx = \int_1^9 (\sqrt{x} - x^{-1/2}) \, dx$$

$$= \frac{2}{3} x^{3/2} - 2 x^{1/2} \bigg|_1^9$$

$$= \frac{2}{3} (3^3) - 2(3) - (\frac{2}{3} - 2)$$

$$= 18 - 6 + 2 - \frac{2}{3}$$

$$= 14 - \frac{2}{3}$$

$$= \frac{40}{3}.$$